Goal: Describe knots, 3-manifolds, 4-manifolds so that we can study and distinguish them

Recall definition:

An n-manifold is a 2^{nd}-countable, Hausdorff space that locally homeomorphic to $\mathbb{R}^n$

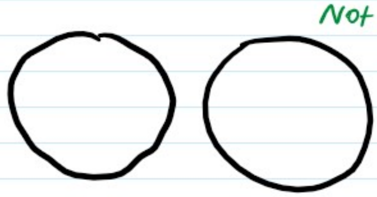
remark: second countable and Hausdorff rule out "bad" examples of topolog. spaces

We'll focus on $n = 1, 2, 3, 4$

Example:

i) $n=1$

S^1 

$S^1 \cup S^1$  Not connected

$\mathbb{R}^1$  Not compact

I  closed interval

Half interval 

Some common adjectives:

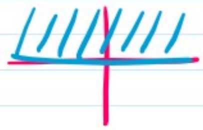
connected

compact   rmk: most of our mfds will be compact

with boundary

↳ every point locally looks like $\mathbb{R}^n$

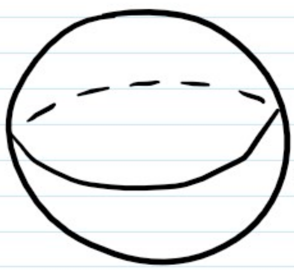
or a neighborhood locally homeomorphic to the closed upper half space of $\mathbb{R}^n$

i.e. 

orientable

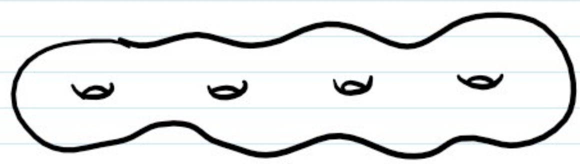
closed = compact & w/out boundary

ii) $n=2$ Let's stick to closed & orientable

sphere  $= \Sigma_0$

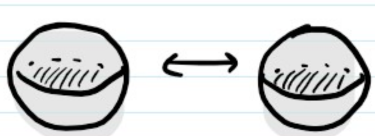
torus  $= \Sigma_1$

2-torus  $= \Sigma_2$

 $= \Sigma_4$

Classification of surfaces: any closed, orientable surface is homeomorphic to one of the above Σ_g

iii) $n=3$ closed orientable

a) 3-sphere $= S^3$  $= B^3 \cup_{S^2} B^3$

↳ unit sphere in $\mathbb{R}^4$

↳ can think of it as a pair of 3-balls B^3 glued to each other along their boundary S^2's

↳ or as $\mathbb{R}^3$ together with a point at ∞

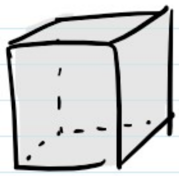
glued to each other along their boundary S^2's

↳ or as $\mathbb{R}^3$ together with a point at ∞

$\mathbb{R}^3 \cup \{\infty\}$

b) 3 torus $= T^3 = S^1 \times S^1 \times S^1$

- cube with opposite faces identified.



c) $S^2 \times S^1$ $\qquad S^1 =$  $=$  line w/ ends identified

↳ so thicken S^2 and then outside & inside spheres are identified

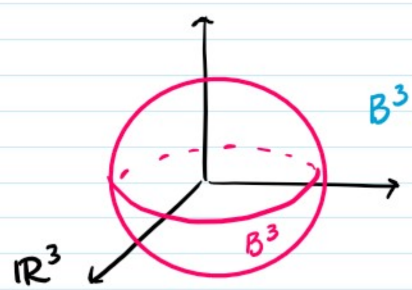
↳ $\mathbb{R}^2$ together with a point at $\infty = S^3$

d) $\Sigma_g \times S^1$

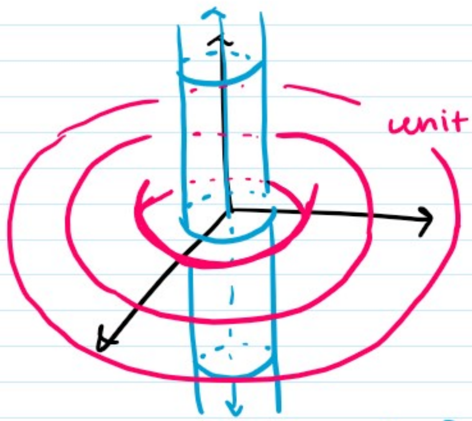
e) Bundles

WAYS TO BUILD 3-MFDS

motivating example: S^3



B^3 = centered at ∞



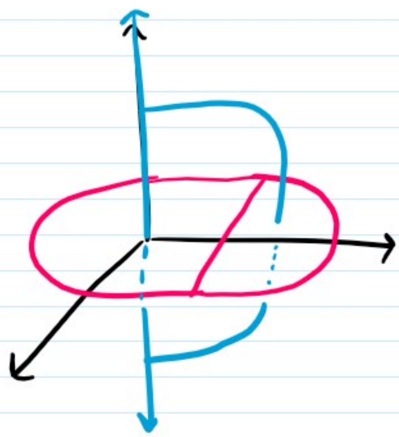
take a closed nbhd of z-axis = solid torus H'

together H and H' give all of S^3 when glued along T^2

$$H \cup_{T^2} H' = S^3$$

Rmk: gluing map $\partial H \longrightarrow \partial H'$ matters!

"Theta graph"

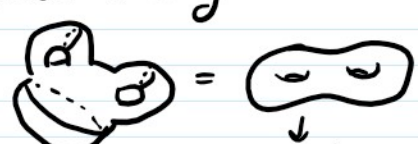


H = nbhd of 

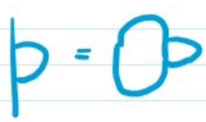
handlebody of genus two

<u>defn:</u>

handlebody = solid tori glued along solid ball B^3



$\hookrightarrow$ also a nbhd of $\bigvee_g S^1$

H' = nbhd of 

"handlebody" of genus 2

claim: $S^3 = H \cup_{\Sigma_2} H'$

glued along Σ_2 = handlebody of genus 2

rmk: all 3-mfds are closed, oriented, connected from now on defn:

A Heegaard splitting of genus g of a 3-manifold Y is a decomposition of Y into two genus g handlebodies glued along their boundaries

Above we had 3 Heegaard splittings of S^3.

Goal: Find other splittings.

Theorem

$\boxed{\text{Any closed oriented 3 mfd admits a Heegaard splitting}}$

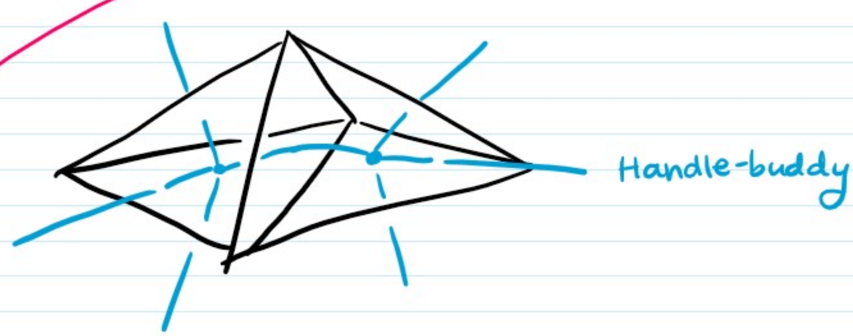
Proof:

Fact: Every 3-mfd admits a triangulation

Take a triangulation of Y

Let H = nbhd of the one-skeleton Y' (all vertices & edges)

Let H' = nbhd of the dual graph



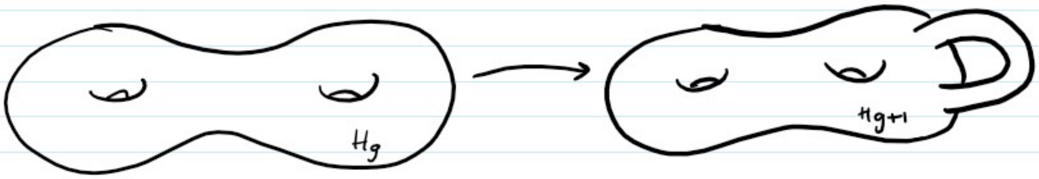
graph whose vertices are centers of the tetrahedra and edges are perpendicular to the faces

Then, $Y = H \cup H'$ ///

Observation: a 3-mfd Y has many different splittings

defn:

A stabilization of a Heegaard splitting $H_g \cup H'_g$ is



add an unknotted 1-handle to H_g to get H_{g+1}

$\parallel$

the handle bounds a disk in Y

$\leftarrow$ disk D^2

Note: $D^2 \times I = B^3$

$$Y = H_g \cup (\text{1-handle} \cup D^2 \times I) \cup H'_g$$

$$= \underbrace{(H_g \cup \text{1-handle})}_{H_{g+1}} \cup \underbrace{(D^2 \times I \cup H'_g)}_{H'_{g+1}}$$

$$= H_{g+1} \cup H'_{g+1}$$

Two Heegaard splittings of Y are equivalent if there exists a homeomorphism of Y taking 1 Heegaard splitting to the other.

Theorem (Reidemeister-Singer)

> Any 2 Heegaard splittings of a 3-mfd Y are stably equivalent

defn: two Heegaard splittings are stably equivalent if they become the same after some number of stabilizations

$Y = H_g \cup_f H'_g$

$f: \Sigma_g \longrightarrow \Sigma_g$ homeom.

$\partial H_g = \Sigma_g = \partial H'_g$

Note: Isotopic homeomorphisms will yield homeom. 3-mfds

↳ homotopic through homeos

So we'd like to study homeomorphisms up to isotopy

defn:

$$\boxed{\mathrm{Mod}(\Sigma_g) = \mathrm{Homeo}^+(\Sigma_g) / \mathrm{Homeo}_0(\Sigma_g)}$$

Homeo^+: orient. preserving homeos

Homeo_0: homeos isotopic to identity

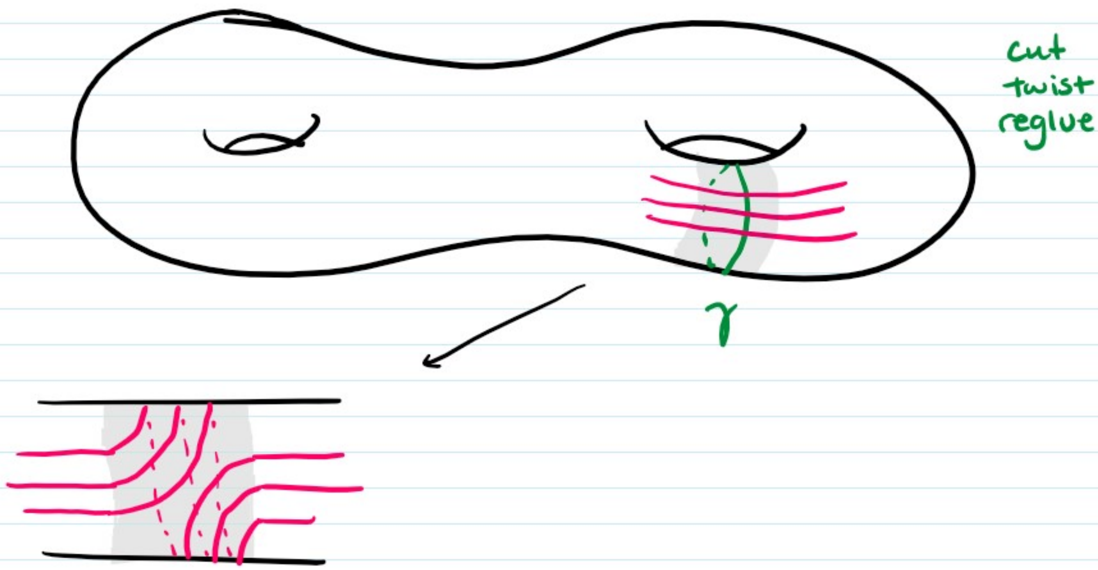
= group of homeos up to isotopy

↳ (identity homeo and composition)

Note: $\mathrm{Homeo}^+(\Sigma_g)$ is an index 2 subgroup of $\mathrm{Homeo}(\Sigma_g)$

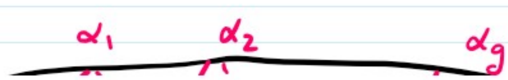
(composition of 2 orientation reversing = orient'n preserving)

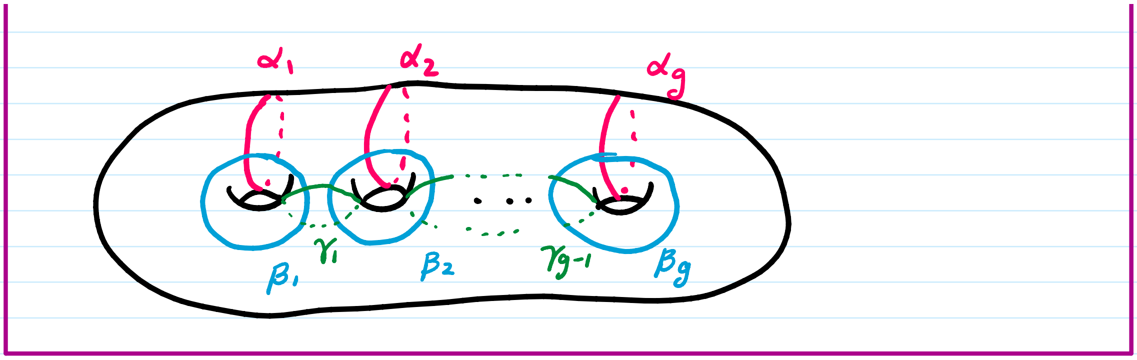
defn: A **Dehn twist** along simple closed curve γ is



Theorem:

$\mathrm{Mod}(\Sigma_g)$ is generated by Dehn twists along α's, γ's, β's.





Theorem:

$$\boxed{\mathrm{Mod}(T^2) \cong SL(2, \mathbb{Z})}$$

integral 2×2 matrices with $\det = 1$

proof:

$$H_1(T^2; \mathbb{Z}) \cong \mathbb{Z} \otimes \mathbb{Z} \qquad (\mu, \lambda)$$

meridian, longitude

Define map $\Pi : \mathrm{Mod}(T^2) \longrightarrow SL(2; \mathbb{Z})$

$$[f] \longmapsto f_* : H_1(T^2) \to H_1(T^2)$$

Exercise: Check that this is a well-defined homeomorphism

fact: any matrix in $SL(2, \mathbb{Z})$ is a product of the matrices of the form $\begin{pmatrix} 1 & \pm 1 \\ 1 & 0 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ \pm 1 & 1 \end{pmatrix}$

$\downarrow$ image of Dehn twists along a meridian μ

$\downarrow$ image of a Dehn twist along longitude λ

Hence Π is surjective.

Π is also injective.

↳ proof: see Rolfsen Theorem 2.D.4

Examples of Heegaard Splittings:

genus 0: $S^3 = B^3 \cup B^3$

rmk: this is the only one

genus 1: $Y = (D^2 \times S^1) \cup_f (D^2 \times S^1)$

requires f to be orientation reversing homeomorphism of T^2

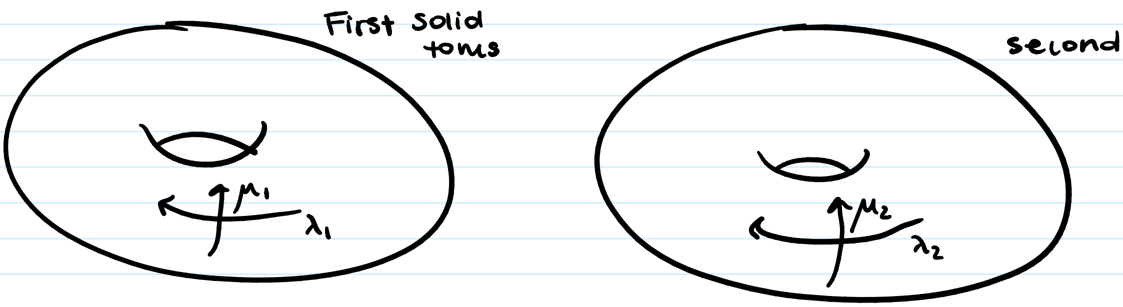
Isotopy classes of orientation reversing homeomorphisms of T^2 are all of the form τA $\quad A \in SL(2, \mathbb{Z})$

$\tau = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\boxed{\begin{pmatrix} -q & s \\ p & r \end{pmatrix} \overset{\text{meridian}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} -q \\ p \end{pmatrix}}$$

i.e. $\begin{pmatrix} -q & s \\ p & r \end{pmatrix}$ with $\det = -1$

$qr + ps = 1$



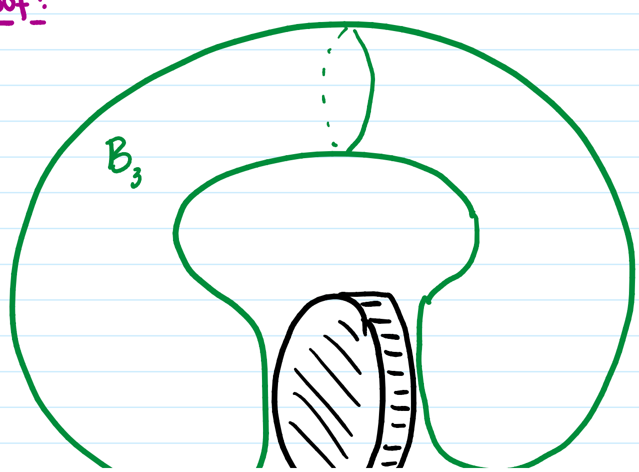
$D^2 \times S^1$ $\qquad$ $D^2 \times S^1$

If we glue by this matrix, then μ_1 is identified w/ $-q\mu_2 + p\lambda_2$ on the second solid torus.

Lemma:

The image of μ_1 determines the 3-mfd Y

proof:



$D^2 \times S^1 = (D^2 \times I) \cup B^3$

Attaching the solid torus can be done in 2 steps.

(I) Attach $D^2 \times I$ (specified by image of μ_1)

(II) Attach 3-ball B^3

$D^2 \times I$

II) Attach 3-ball B^3

Fact: any orientation preserving homeo of S^2 is isotopic to identity

$$\pi(f) = A = \begin{pmatrix} -q & s \\ p & r \end{pmatrix}$$

$qr + ps = 1 \Rightarrow (p, q)$ relatively prime

$$Y = (D^2 \times S^1) \cup_f (D^2 \times S^1)$$

from previous lemma, Y is determined by p, q

defn: we call this a lens space $L(p,q)$

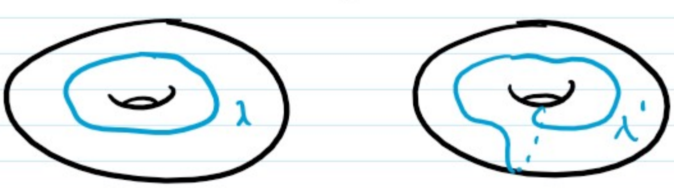
WLOG: can choose $p \geq 0$

(reversing orientations of μ_1 and λ_1 sends A to $-A$)

$D^2 \times S^1$

Note: meridians are uniquely determined up to isotopy and orientation as the curve that bounds a disk in the solid torus

· longitudes are not unique



$\hookrightarrow$ it should intersect meridian exactly once

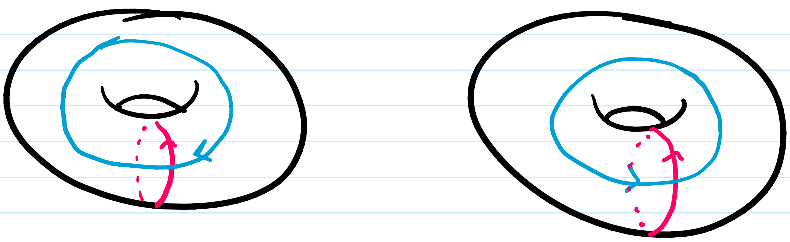
$n\mu_1 + \lambda_1$ is also a longitude.
↳ adds n times 1^{st} column of A to 2^{nd} column

$n\mu_2 + \lambda_2$ is also a longitude
↳ subtracts n times 2^{nd} row of A to the first row) $\Big\}$ $(*)$

<u>$p=0$</u> $A = \begin{pmatrix} -q & s \\ 0 & r \end{pmatrix}$

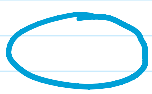
claim: WLOG $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$L(0,1)$



$S^2 \times S^1$.

become an S^2 and we see an



S^1's worth of them.

If $p \neq 0$, WLOG you can choose $0 \leq q \leq p-1$ (use $*$)

$p=1$ WLOG $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$L(1,0) = S^3$ $\downarrow$
glue μ_1 to λ_2 and λ_1 to μ_2.

$p \geq 2$ $L(p,q)$

$L(p,q)$ $L(p',q')$

Question: When are two lens spaces homeomorphic?

Exercise: Compute $H_1(L(p,q); \mathbb{Z})$

Exercise: Compute $H_1(L(p,q);\mathbb{Z})$

partial answer: need $p = p'$ — a.k.a. $qq^{-1} \equiv 1 \pmod p$

$L(p,q) = -L(p,-q)$

Exercise: $L(p,q) \cong_{\text{homeo}} L(p',q') \iff q' = \pm q^{\pm 1} \pmod p$

Fact: $L(p,q) \sim_{\text{homot. equiv.}} L(p',q') \iff qq' = \pm m^2 \pmod p$ for some $m \in \mathbb{Z}^2$