Week 13, 11/14 & 11/16

$S$ bilinear form

$S: H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$

$S(v,w) = v^T S w$

say $U$ is a change of basis matrix

$w = U w'$

$v = U v' \qquad v^T = (v')^T U^T$

$v^T S w = (v')^T U^T S U w'$

In contrast:

$A$ is a linear map

$A: V \longrightarrow V$

$A(v) = w$

$U$ change of basis:

$v = U v'$

$w = U w'$

$A U v' = U w'$

$U^{-1} A U v' = w'$

---

## Alexander module and infinite cyclic covers:

(Rolfsen Ch. 7)

$X = S^3 - \nu(K)$

$H_1(X; \mathbb{Z}) = \mathbb{Z}$

$G = \ker\left(\pi_1(X) \longrightarrow H_1(X)\right)$

$\tilde{X}$ covering space of $X$ corresp. to subgroup

by construction is normal,

so deck group is $\mathbb{Z}$ and generated by $t$

by construction is abelian,
so deck group is $\mathbb{Z}$ and generated by $t$

So covering space has action of $t$.

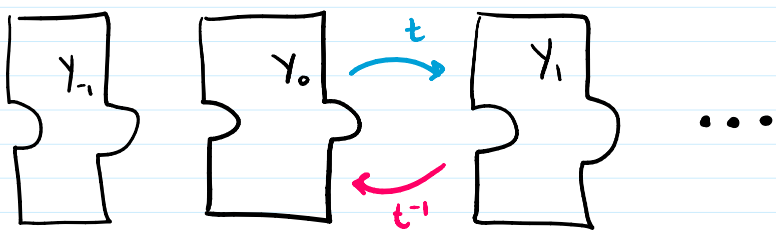
$H_1(\tilde{X})$ also has an action by $t$

so $H_1(\tilde{X})$ is a $\mathbb{Z}[t, t^{-1}]$-module

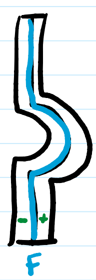
from Abelian — from deck group.

Let $Y$ be $S^3 - F$ $\quad F$ = Seifert Surface

schematically: $\quad Y_i$ is a copy of $Y$




the way they're glued together by a copy of $F$,




We cut along the Seifert surface because meridian in $K$ is a loop in $X$ but not a loop in $\tilde{X}$, so this cut ensures this.

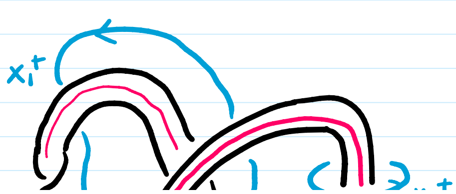
this whole thing is $\tilde{X}$.

In general, $H_1(\tilde{X})$ may not be finitely generated as an Abelian group (i.e. as a $\mathbb{Z}$-module).

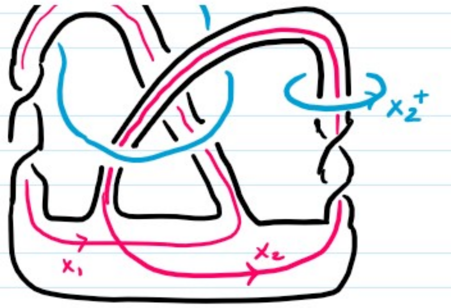
However, it is finitely generated as a $\mathbb{Z}[t, t^{-1}]$-module.

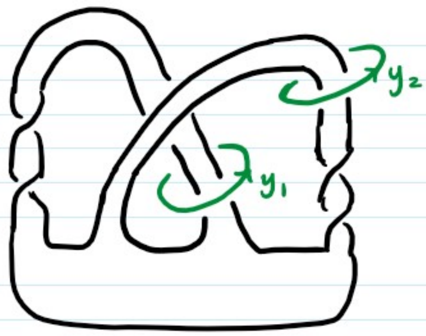
**Example:**

LHT



LHT




$y_1, y_2$ are the generators for $H_1(S^3 - F)$

Now we can write $x_i^+$ in terms of $y_1$ and $y_2$.

$$x_1 \xrightarrow{+} y_1 - y_2$$

$$x_2 \xrightarrow{+} y_2$$

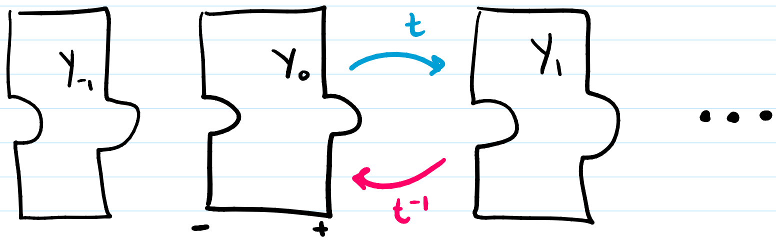
Could also push off in the minus direction

Exercise: check that $x_1 \xrightarrow{-} y_1$

$$x_2 \xrightarrow{-} -y_1 + y_2$$

So, $y_1 = t(y_1 - y_2)$

$$-y_1 + y_2 = t(y_2)$$




$$y_1 - y_2 \xrightarrow{t} y_1$$

$$y_2 \xrightarrow{t} -y_1 + y_2$$

so, $\quad y_1 = y_2 - ty_2$

in particular, we only needed $y_2$

$$y_2 - ty_2 = t(y_2 - ty_2 - y_2)$$

$$y_2 - ty_2 = -t^2 y_2$$

$$t^2 y_2 - ty_2 + y_2 = 0$$

If we want to describe $H_1$ of the infinite cyclic cover,

$$H_1(\tilde{X}) = \langle t^i y_1, t^i y_2 \mid t^i y_1 = t^{i+1}(y_1 - y_2),\ t^i(-y_1 + y_2) = t^{i+1} y_2 \rangle$$

$$\Big\downarrow \text{ simplifies to (set } y = y_2)$$

$$H_1(\tilde{X}) = \langle t^i y \mid t^2 y - ty + y \rangle$$

$$H_1(\tilde{X}) = \mathbb{Z}[t, t^{-1}] / \langle t^2 - t + 1 \rangle$$

$\uparrow$

Alexander polynomial of the trefoil

Alexander module of a polynomial as a $\mathbb{Z}[t,t^{-1}]$-module

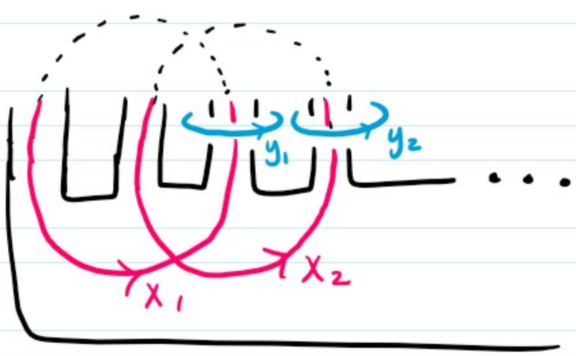
**Exercise:**

Show that for a Seifert matrix $S$, $S - tS^T$, or equivalently $S^T - tS$, is a presentation for the Alexander module

$\Lambda = \mathbb{Z}[t,t^{-1}]$ $\qquad \Lambda^{2g} \longrightarrow \Lambda^{2g}$

<u>hint:</u> Consider basis $\{x_i\}$ for $H_1(F)$ and a dual basis $\{y_i\}$ for $H_1(S^3 - F)$

i.e. $lk(x_i, y_j) = \delta_{ij}$




Recall in Seifert form,

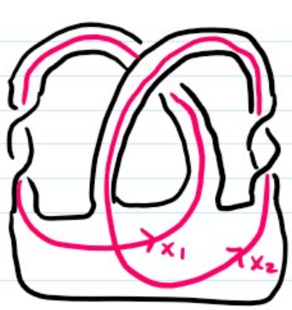
$lk(x_i, x_j^+) = lk(x_i^-, x_j)$

The intersection form $I$ on $H_1(F)$ is

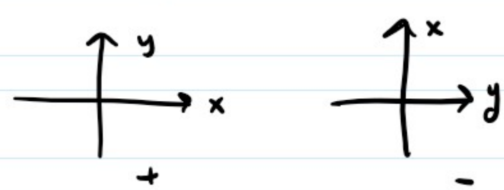
$$I : H_1(F) \times H_1(F) \longrightarrow \mathbb{Z}$$

$$I(x,y) = x \cdot y$$

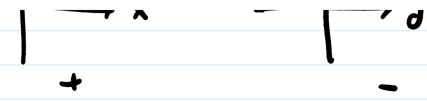
signed intersection count




$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$



**Exercise:** Check that $I$ is skew-symmetric

**Lemma:**

Let $F$ be a closed oriented surface and $I$ its intersection form with respect to a chosen basis. Then, there exists a real matrix $U$ such that $I = U^T J U$ where $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \oplus \dots \oplus \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

**proof:** see Sabeliev Lemma 7.6.

**Corollary**

$$\boxed{\det(I) = 1}$$

Q: what matrices can be Seifert forms?

**Exercise:**

① $I = S^T - S$

② $\Delta_K(t^{-1}) \doteq \Delta_K(t)$

Recall: equivalent up to factor of $t$

**Corollary**

$$\boxed{\Delta_K(1) \doteq 1}$$

**proof:** $\Delta_K(1) = \det(S^T - S)$

## Other invariants from Seifert form:

$Q = S + S^T$ symmetric

**knot determinant:** $|\det Q| = |\Delta_K(-1)|$

$\det(Q)$ is always odd

**knot signature** $\sigma(K) = \mathrm{sign}(Q)$

exercise: check that this is a well-defined knot invariant

Exercise:

check if $S$ is a Seifert matrix for $K$, then $-S$ is the Seifert matrix for mirror $mK$

Example:

1. LHT $S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$

$Q = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\lambda = 1, 3$

2 positive eigenvalues

$\sigma(LHT) = 2$

2. RHT $S = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}$ $\sigma(LHT) = -2$

Exercise:

- $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$
- $\sigma(mK) = -\sigma(K)$
- $\sigma(K)$ is always even

## Levire-Tristam signatures

$w \in \mathbb{C} \quad |w| = 1$

motivation: (want Hermitian)

$$Q_w = \tfrac{1}{2}(1-w)S + \tfrac{1}{2}(1-\bar{w})S^T$$

Recall: $Q = S + S^T$

so, $Q_{-1} = S + S^T$ is what we did earlier

$$\sigma_w(K) = \operatorname{sign} Q_w$$

Remark: $Q_w$ may be singular so define

$$Q_w(K) = \frac{\lim_{\alpha \to w^+} \operatorname{sign} Q_\alpha + \lim_{\alpha \to w^-} \operatorname{sign} Q_\alpha}{2}$$

## Fibered knots:

A map $f: E \longrightarrow B$ is a fibration with fiber $F$ if each point in the base $B$ has a neighborhood $U$ and a trivializing homeomorphism $h: f^{-1}(U) \longrightarrow U \times F$ such that the following diagram commutes:

$$\begin{tikzcd}
f^{-1}(U) \arrow[rr, "h"] \arrow[dr, "f"'] & \arrow[d, phantom, "\circ"] & U \times F \arrow[dl, "proj"] \\
& U &
\end{tikzcd}$$

commonly written

$$\begin{tikzcd}
F \arrow[r] & E \arrow[d, "f"] \\
& B
\end{tikzcd}$$

Remark: $f^{-1}(b) \cong F \quad \forall\, b \in B$

**defn:**

A knot $K \subset S^3$ is a fibered knot if there is a fibration $f: S^3 - K \longrightarrow S^1$ such that $f$ is a well-behaved near $K$

ie. $K$ has a neighborhood $S^1 \times D^2$ such that

① $K$ is $S^1 \times \{0\}$

② $f: S^1 \times (D^2 \times \{0\}) \longrightarrow S^1$

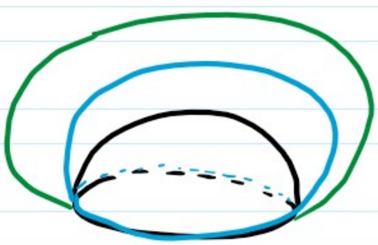
$$(x, y) \longmapsto y/|y|$$

**Exercise:**

Let $K$ be a fibered knot. The closure of a fiber is a Seifert surface for $K$

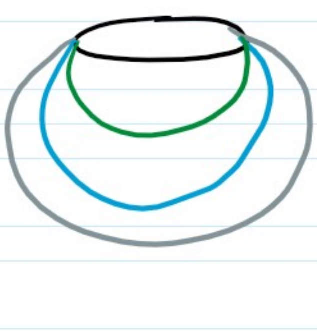
**Example:** the unknot is fibered

disk = fiber



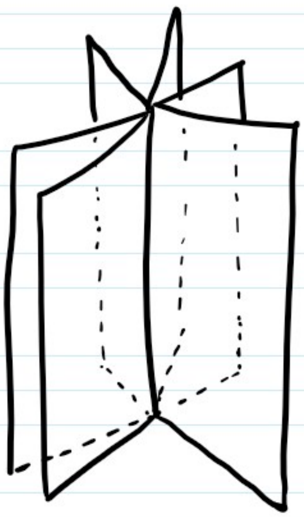
bubbling disks getting bigger

disk @ $\infty$



disks bubble back to

Could also send a point on unknot to $\infty$ and see the $S^1$'s worth of disks:



## Example:

1.  is fibered. Rolfsen Ch. 10.I

2.  is fibered. (figure 8)

## Non-examples:

1. Most twist knots for most $n$ is not fibered

$K_n =$ 


i.e. fibered when

$K_n$ = unknot, trefoil, figure 8

Not fibered otherwise.

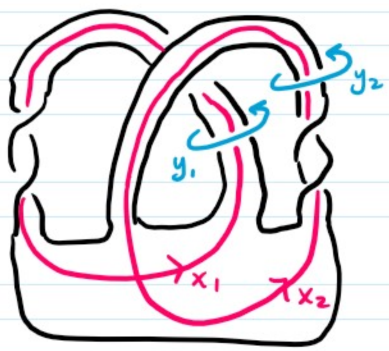
A locally trivial bundle $f: E \longrightarrow S^1$ can be described as follows:

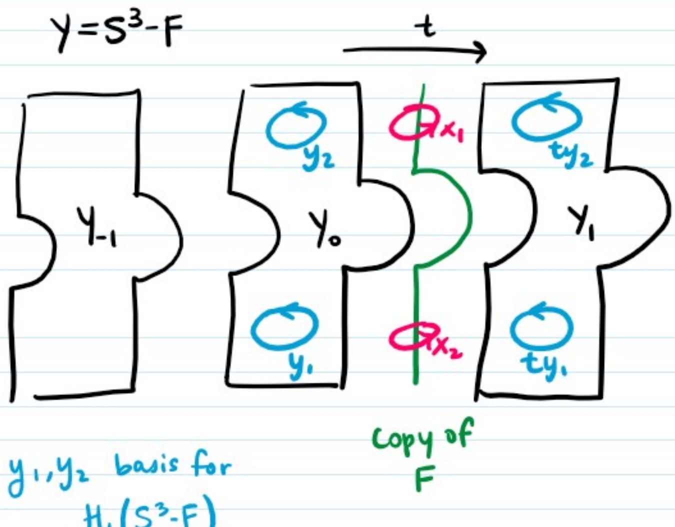
$$S^1 = [0,1] / \{0\} \sim \{1\}$$

$[0,1]$ is contractible so every bundle over $[0,1]$ with fiber $F$ is just $F \times [0,1]$

The bundle $E$ is obtained by identifying $F \times \{0\}$ and $F \times \{1\}$ by a homeomorphism of $F$ to itself called the monodromy

Example (from before)



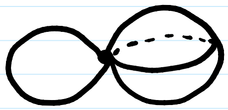

$$y_1 \xleftarrow{-} x_1 \xrightarrow{+} y_1 - y_2$$

$$-y_1 + y_2 \xleftarrow{-} x_2 \xrightarrow{+} y_2$$

$$y_1 = t(y_1 - y_2)$$

$$-y_1 + y_2 = t(y_2)$$

Example:

$X =$ 

$\tilde{X} =$  infinite cyclic cover

$H_2(\tilde{X})$ as $\mathbb{Z}[t,t^{-1}]$ (you just need one sphere)
is $\mathbb{Z}[t,t^{-1}]\langle y\rangle$

$H_2(\tilde{X})$ as an Abelian group (it's infinitely generated)
is $\overset{\infty}{\oplus}\mathbb{Z}$

**Exercises:**

up to a factor of $t^{\pm 1}$

Recall that $\Delta_K(1) \doteq 1$ and
$\Delta_K(t) \doteq \Delta_K(t^{-1})$

① Show that any degree (difference of largest and smallest power of $t$) 2 polynomial satisfying these conditions can be realized as $\Delta_K(t)$ for some $K$

② **bonus:** Show that any polynomial $f(t)$ with $f(1) \doteq 1$ and $f(t) = f(t^{-1})$ can be realized as $\Delta_K(t)$

## Fibered Knots continued . . .

$$\begin{tikzcd} F \arrow[r] & S^3 - K \arrow[d, "f"] \\ & S^1 \end{tikzcd}$$

$h: F \longrightarrow F$ homeomorphism

$$S^3 - K = F \times I / (x,0) \sim (h(x),1)$$

$h: F \longrightarrow F$ homeomorphism called the monodromy

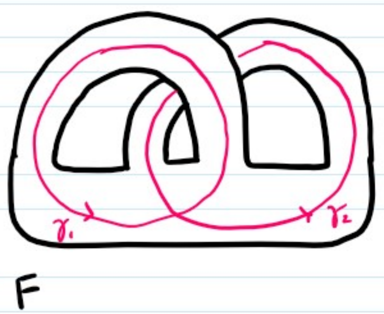
Example:

$K =$ unknot

$F =$ disk

$h:$ identity since $S^3 - K = D^2 \times S^1$

Example genus 1 fibered knots




$K =$ figure eight

$h = \tau_{\gamma_1} \circ \tau_{\gamma_2}^{-1}$

$K =$ RHT

$h = \tau_{\gamma_1} \circ \tau_{\gamma_2}$

homological monodromy:

$$h_* : H_1(F) \longrightarrow H_1(F)$$

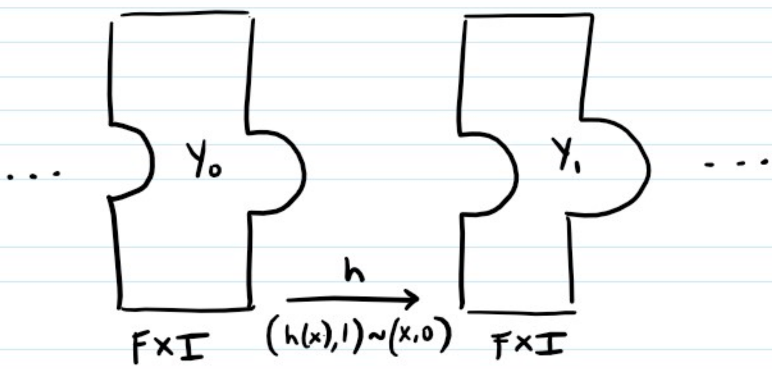
**Proposition:**

Let K be a fibered knot with fiber F. Choose a basis for $H_1(F)$ and let M be the homological monodromy with this basis. Then $M - tI$ is a presentation matrix for Alexander module and hence

Let $K$ be a fibered knot with fiber $F$. Choose a basis for $H_1(F)$ and let $M$ be the homological monodromy with this basis.
Then $M - tI$ is a presentation matrix for Alexander module and hence

$$\Delta_K(t) = \det(M - tI)$$

proof sketch:




$M = tI$

///

In fact,

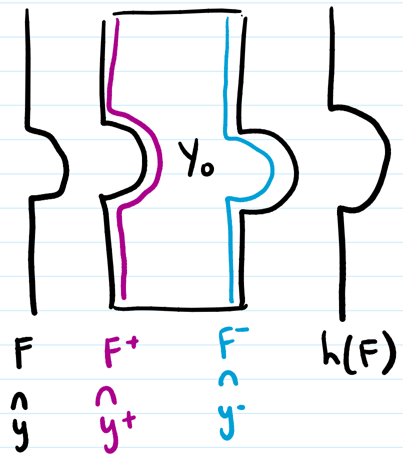
Lemma

Let $K$ be a fibered knot. Fix a basis for $H_1(F)$. Let $S$ be the Seifert matrix and $M$ the homological monodromy with respect to this basis.
Then $M^T S = S^T$

proof:

Need to show $\quad x^T M^T S y = x^T S^T y$ transpose of a number

$$= y^T S x$$




$S(x,y) = lk(x, y^+)$

$Mx = h_*(x)$

Need to show $\underbrace{lk(h_t(x), y^+)}_{lk(x,y^-)} = lk(y, x^+)$

///

**Example:**

$K = LHT$

$$S = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$$

$$M = (S^T)^{-1} S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix}$$

**Corollary**

K fibered with fiber F.

$$g(F) = g(K)$$

**proof:**

$$g(F) = \tfrac{1}{2} \deg \Delta_K(t) \leq g(K) \leq g(F)$$

$$\Delta_K(t) = \det(M - tI)$$

**Corollary:**

$\boxed{\text{If } K \text{ is fibered, then } \Delta_K(t) \text{ is monic.}}$

first coefficient is $\pm 1$, equivalently last coefficient is $\pm 1$

**proof:**

$$\Delta_K(t) = \det(M - tI)$$

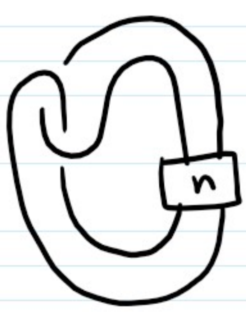
$$\Delta_K(0) = \det(M)$$

$M$ is invertible since $h$ is a homeomorphism hence

$$\det(M) = \pm 1.$$

So it can tell us when a knot is not fibered.

**Example:** twist knot $K_n$




$$\Delta_{K_n}(t) = nt^2 - (2n+1)t + n$$

$K_n$ is not fibered for $n \neq \pm 1, 0$

($n = \pm 1, 0$ is exactly unknot, trefoil, figure 8)

**Why HFH rocks:**

- $\Delta_K(t)$ obstructs fibering and gives lower bound for $g(K)$

- Heegaard floer homology detects fibering and gives you $g(K)$

**Open Q's:**

① $c(K_1 \# K_2) = c(K_1) + c(K_2)$ ?

crossing number

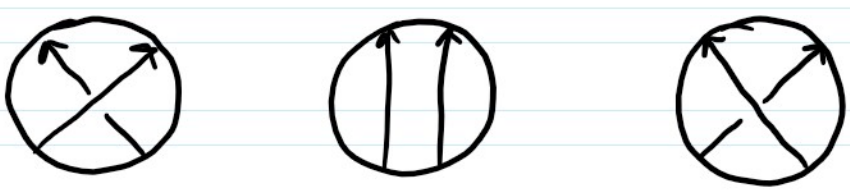
② $u(K_1 \# K_2) = u(K_1) + u(K_2)$ ?

unknotting number

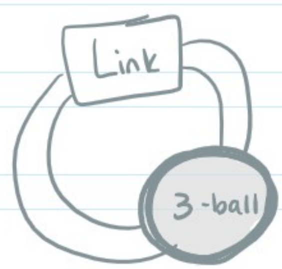
## Skein Relations:

**Theorem** (Conway)

Let $L_+, L_0, L_-$ be 3 oriented links which coincide away from a ball $B$ and intersect $B$ in two unknotted arcs



$L_+$ $\qquad$ $L_0$ $\qquad$ $L_-$



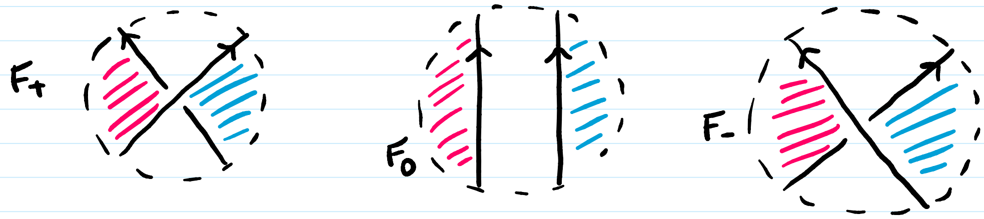

then $\Delta_{L_+}(t) - \Delta_{L_-}(t) + (t^{1/2} - t^{-1/2})\Delta_{L_0}(t) = 0$

**Remark:**

This requires/dictates a certain normalization

**Idea of Proof:**

## Idea of Proof:




$L_+$ $\qquad$ $L_0$ $\qquad$ $L_-$

$\{x_1, \dots, x_n\}$ basis for $H_1(F_0)$

$x_0$ curve on $F_\pm$ s.t. $\{x_0, x_1, \dots, x_n\}$ basis for $H_1(F_\pm)$

$S_0$ seifert matrix for $F_0$

$$S_- = \begin{pmatrix} * & * & \cdots & * \\ * & & & \\ \vdots & & S_0 & \\ * & & & \end{pmatrix}$$

$$S_- = S_+ + \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix}$$

Alexander polynomial for $L_-$

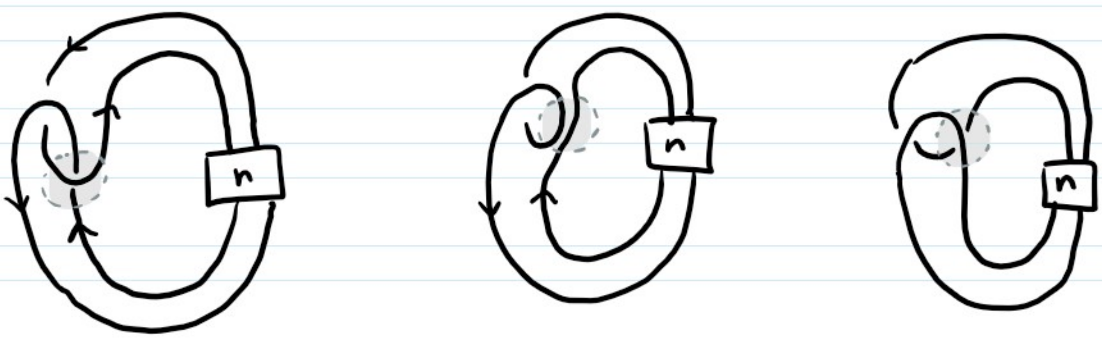
$$t^{1/2} S_- - t^{-1/2} S_-^T = t^{1/2} S_+ - t^{-1/2} S_+^T + \begin{pmatrix} t^{1/2} - t^{-1/2} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & 0 & \\ 0 & & & \end{pmatrix}$$

$$= \begin{pmatrix} * & \cdots & * \\ \vdots & t^{1/2}S_0 - t^{-1/2}S_0^{T} & \\ * & & \end{pmatrix}$$

**Exercise:**

check that expanding $\det\left(t^{1/2}S_{\pm} - t^{-1/2}S_{\pm}\right)$ along 1st column yields desired result.

**Example:** (with twist knots)




$L_+$ $\qquad$ $L_0$ $\qquad$ $L_-$

$L_0$ = bounds annulus w/ n full twists

$$\Delta_{L_0} = n\left(t^{1/2} - t^{-1/2}\right)$$

$L_-$ = unknot

$$\Delta_{L_1} = 1$$

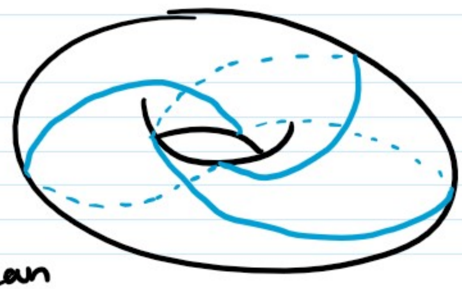
$$\Delta_{L_+}(t) - 1 + \left(t^{1/2} - t^{-1/2}\right)\cdot\left(n\left(t^{1/2} - t^{-1/2}\right)\right)$$

$$\therefore\ \Delta_{L_+}(t) = nt + (2n+1) - nt^{-1}$$

# Knot Group:

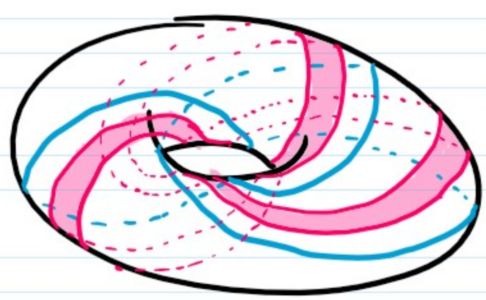
$\pi_1(S^3 - K)$

**Example:**

$K =$  

is a torus link b/c it can be embedded on $T^2$.

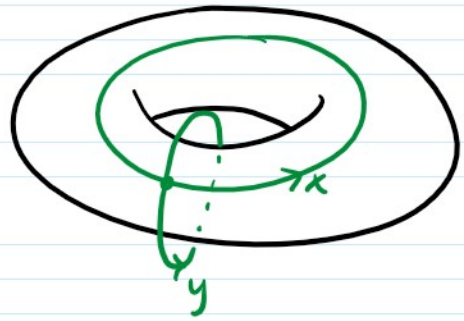
$S^3 - K$ = union of two solid tori glued along annulus $A = T^2 - K$



$T_{2,3}$

longitude ↑ ↑ meridian

core of annulus runs around $x$ twice and around $y$ thrice




Seifert Van-Kampen

$$\pi_1(S^3 - K) = \langle x, y \mid x^2 = y^3 \rangle$$

**Claim:** $G$ is not Abelian

$K$ is non-trivial

**proof:** homomorphism of free group $\langle x, y \rangle$ to $S_3$:

$x \longmapsto (12)$

$y \longmapsto (123)$

induces a homomorphism $G \longrightarrow S_3$

(descends because $(12)^2 = (123)^3$)

$(12)$ and $(123)$ generate all of $S_3$, which is not Abelian.

(so original group is not Abelian)

Hence $G$ is non Abelian.

//

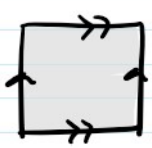
**Alternate proof:**

$$G \xrightarrow{\text{surjective}} G / \langle x^2 = 1 \rangle \cong \mathbb{Z}/2 * \mathbb{Z}/3$$

///

**Exercise:**

$T_{p,q}$ is nontrivial



$p, q$ relatively prime to be a torus knot

$T_{p,q}$ = (p,q) torus knot