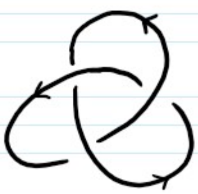
Week 14, 11/21

Tuesday, November 21, 2023 12:30 PM

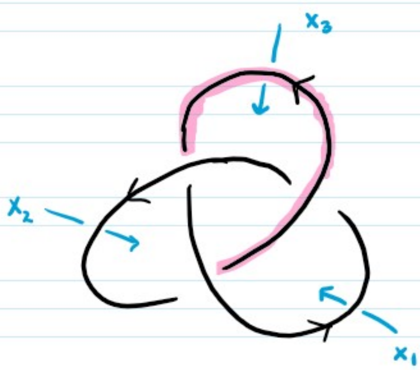
Last time:

- fibered knots
- skein relation
- knot group

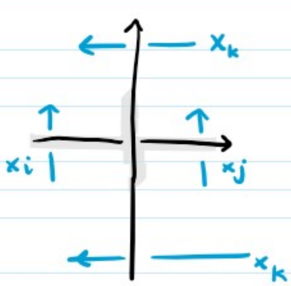
Wirtinger Presentation



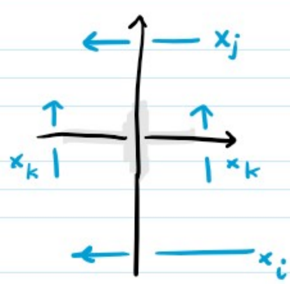
each arc in diagram gives a generator



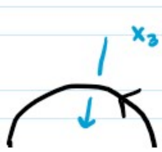
At each crossing we get a relation

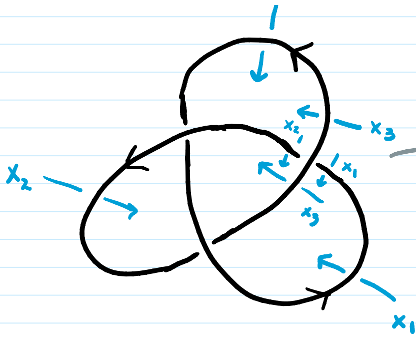


$x_k x_i = x_j x_k$



$x_k x_j = x_i x_k$





$\longrightarrow x_3 x_2 = x_1 x_3$

So presentation is

$$\langle x_1, x_2, x_3 \mid \underset{①}{x_3 x_2 = x_1 x_3},\ \underset{②}{x_2 x_1 = x_3 x_2},\ \underset{③}{x_1 x_3 = x_2 x_1} \rangle$$

Note:
- ③ is a result of ① and ②
- only Abelian presentation is that of unknot
- Abelianization of group is always $\mathbb{Z}$

generalization:

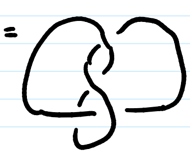
Remark: Any one of the relations can be omitted

Exercises:

① Check that this group is isomorphic to $\langle x, y \mid x^2 = y^3 \rangle$

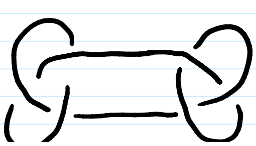
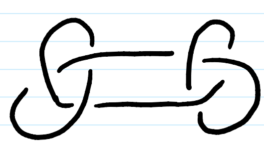
<u>hint:</u> $x_3 = x_1^{-1} x_2 x_1$ so only need $\langle x_1, x_2 \mid x_1^{-1} x_2 x_1 x_2 = x_2 x_1 \rangle$

② Compute the knot group for the figure eight:

$\pi_1(S^3 - K)$ for $K =$ 

③ Use $\pi_1(S^3 - K)$ to show that  and  are distinct

④ Show that

 and 

and

Square knot

LHT # RHT

granny knot

LHT # LHT

have isomorphic knot groups

Remark: Each generator in Wirtinger presentation is a meridian

Exercise:

Any two meridians are conjugate to one another

e.g.

$x_3 = x_1^{-1} x_2 x_1 \quad \therefore \quad x_3$ is a conjugate of x_2

defn: the weight of a group G is the smallest # w such that there exist $g_1, \dots, g_w \in G$ with the property that the normal closure of $\{g_1, \dots, g_w\}$ is G

Recall: the normal closure of S in G

$$ncl_G(S) = \{ g_1^{-1} s_1^{\varepsilon_1} g_1 \cdots g_n^{-1} s_n^{\varepsilon_n} g_n \mid n \geq 0, \ \varepsilon_i = \pm 1, \ s_i \in S, \ g_i \in G \}$$

"smallest normal subgp in G containing S"

Exercise:

Any knot group $\pi_1(S^3 - K)$ for any K has weight 1 since $\pi_1(S^3 - K)$ is normally generated by a meridian

Can we add any extra info to make $\pi_1(S^3 - K)$ a complete invariant?

Peripheral Subgroup:

Consider $\partial(\nu K) = T^2 \subset S^3 - \nu(K)$

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T^2

The **peripheral subgroup** of $\pi_1(S^3 - \nu(K))$ is the conjugacy class of $\pi_1(T^2) \subset \pi_1(S^3 - \nu(K))$

Note: $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ generated by a meridian m and a (0-framed) longitude ℓ

The data of m and ℓ is called a **peripheral system**

Theorem: (Fox)

No isomorphism of the knot groups of the square knot and the granny knot respects peripheral subgroups

Theorem: (Waldhauser)

The knot group together with a peripheral system is a complete knot invariant

yay! But hard to work with

What else is a peripheral system good for?

A peripheral system can help us to compute $\pi_1(S^3_{p/q}(K))$

$$\pi_1(S^3_{p/q}(K)) \cong \pi_1(S^3 - K) / m^p \ell^q$$

Exercise:

the quotient of a weight 1 group is at most weight 1

Hence, $\pi_1(S^3_{p/q}(K))$ is weight at most 1

defn: A group G is called **perfect** if it equals its commutator subgroup

Equivalently, G is perfect if its Abelianization is trivial

Examples:

① $MCG(S_g)$ $g \geq 2$ is perfect

② $\pi_1(PHS)$ is perfect since $H_1(PHS) = 0$

③ $\pi_1(\mathbb{Z}HS^3)$ is perfect

④ $S^3_{1/n}(K)$ $n \in \mathbb{Z} \setminus \{0\}$ is a $\mathbb{Z}HS^3$

defn: The **Dehn surgery number** $S_D(Y)$ is the minimal number of link components in a surgery description of Y

Example:

$S_D(PHS) = 1$ (surgery on RHT)

Exercise:

Use H_1 to show that $S_D(\#_n \mathbb{R}P^3) = n$ — $L(2,1)$

hint: get an upper and lower bound

$$H_1(\#_n \mathbb{R}P^3) = (\mathbb{Z}/_2)^n$$

needs n generators

Remark:

Work in progress of Daemi-Miller-Eismeir show that

$S_D(\#_n \, PHS) = n$

(proof relies on guage theory)

Question: Can π_1 help us study S_D ?

If weight of $\pi_1(Y) \geq n$, then it follows that $S_D(Y) \geq n$.

$\boxed{\text{Wiegold Conjecture:}}$

$\boxed{\text{Every perfect group has weight one}}$

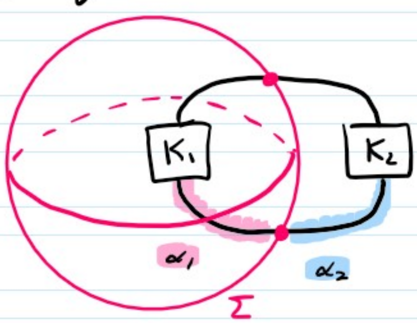
$\boxed{\text{Theorem:}}$

$$\boxed{g(K_1 \# K_2) = g(K_1) + g(K_2)}$$

proof: $g(K_1 \# K_2) \leq g(K_1) + g(K_2)$ clear via connect sum of Seifert surfaces.

Now let F be a minimal genus Seifert surface for $K_1 \# K_2$

Let Σ be a 2-sphere intersecting $K_1 \# K_2$ in two points (coming from the connect sum)



$\beta =$ arc in Σ joining $\Sigma \cap (K_1 \# K_2)$

$\alpha_i \cup \beta = K_i$

General position argument:

$F \cap \Sigma$ is a 1-manifold

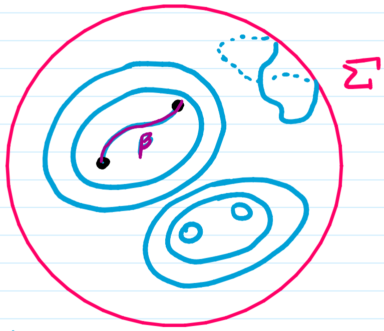
i.e. finite collection of simple closed curves

and an arc β joining 2 points in $\Sigma \cap (K_1 \# K_2)$

$F \cap \Sigma$
$=$ a bunch of circles and an arc

↓

w/ boundary the two points in $\Sigma \cap (K_1 \# K_2)$



F = minimal genus Seifert surface for $K_1 \# K_2$

$\boxed{\text{2-D Schönflies theorem:}}$

$\boxed{\text{Any smooth simple closed curve separates } S^2 \text{ into 2 disks}}$

let C be a s.c.c. of $F \cap \Sigma$ that is innermost on $\Sigma - \beta$

$\Rightarrow$ C bounds a disc D,

the interior of which misses F

$\Rightarrow$ create $\hat{F}$ by deleting a small annular nbhd of C and replacing it by two disks, each a parallel copy of D

If C were non-separating, then $\hat{F}$ would have smaller genus than F, a contradiction because F is assumed to be the minimal.

$\Rightarrow$ C must be separating and $\hat{F}$ is disconnected

Consider the component of $\hat{F}$ which contains $K_1 \# K_2$

Has the same genus as F but meets Σ in fewer s.c.c.'s (C at least has been eliminated)

Repeat until we obtain F' that only intersects F in β, so Σ separates F' into Seifert surfaces for K_1 and K_2

Hence $\quad g(K_1) + g(K_2) = g(K_1 \# K_2)$

///

$(\{\text{knots in } S^3\}, \#)$, a group?

identity: unknot

inverses: No!

proof: $g(K_1 \# K_2) = g(K_1) + g(K_2)$

always non-negative

$g(K) = 0 \iff$ unknot

$\Rightarrow$ no nontrivial knot has an additive inverse

Next time: concordance

an equivalence relation (weaker) on knots

maybe we can get inverses?