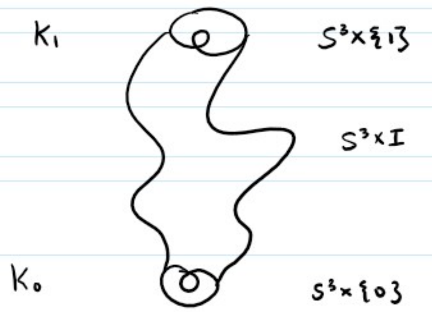
Week 15, 11/28 and 11/30

Tuesday, November 28, 2023 8:26 AM

defn: two knots K_0 and K_1 are **concordant** denoted $K_0 \sim K_1$ if they cobound a smooth, properly embedded annulus A in $S^3 \times [0,1]$

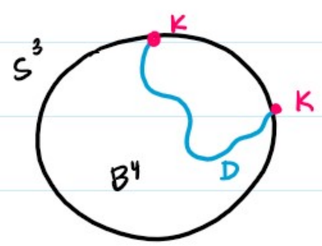


Exercise:

this is an equivalence relation

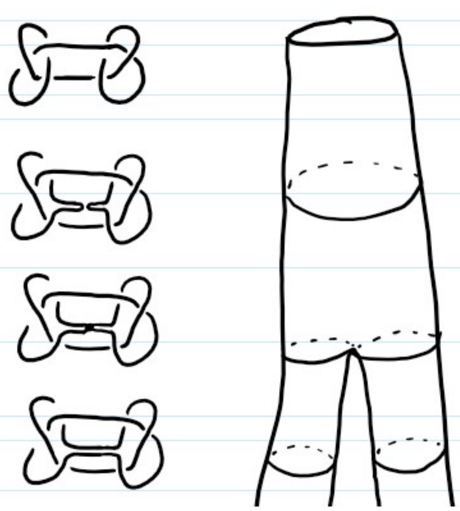
defn: A knot $K \subset S^3$ is **slice** if K bounds a smoothly embedded disk in B^4

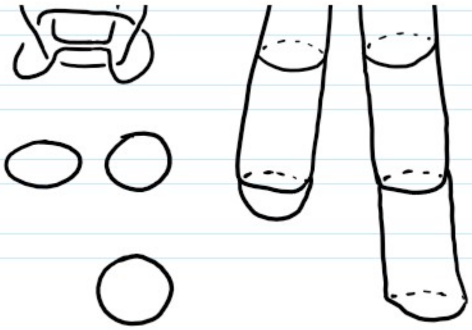
Exercise: K is slice $\iff$ $K \sim$ unknot



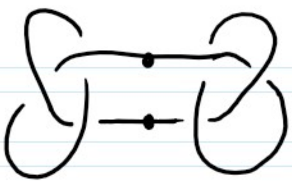
Example:

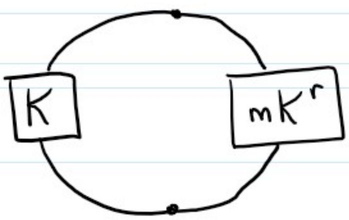
The square knot is slice:





Another way to see this knot is slice:





reverse of mirror image

Let $-K = mK^r$

$\Rightarrow K \# mK^r$ is slice

Knot concordance group

$$\mathcal{C} = \left(\{\text{knots in } S^3\}/\sim \,, \# \right)$$

$$-[K] = [-K]$$

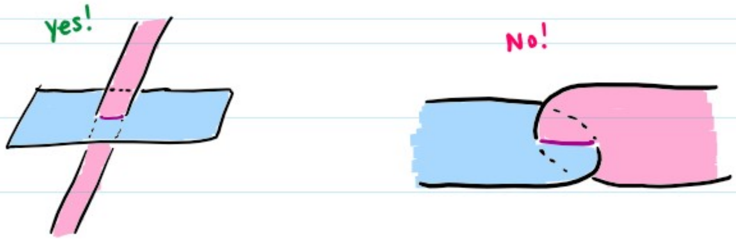
identity = unknot or any slice knot

def'n:

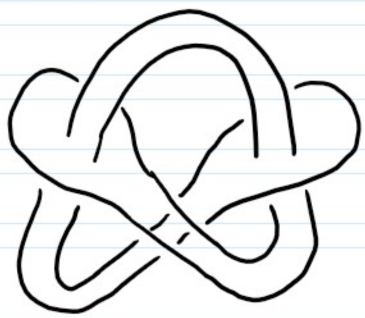
① A knot K is called **ribbon** if K bounds a smoothly embedded disk in B^4 with no local maxima w.r.t. the radial Morse function

② Equivalently, if K bounds an immersed disk in S^3 with only ribbon singularities

defn: pre-image of any arc of self intersection is two arcs in D^2, one of which is interior

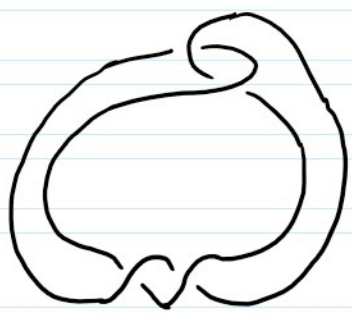


Example:



ribbon knot

Nonexample:



this surface is not ribbon

(this knot is not slice)

proof that defn ② $\Rightarrow$ defn ① of ribbon

With an extra dimension, push the part of the disk with the interior self-intersection deeper into B^4

① $\Rightarrow$ ②

Exercise: hint: saddle points give bands

Remark: K is ribbon $\Longrightarrow$ K is slice

Slice-Ribbon conjecture

A knot is slice $\Longleftrightarrow$ it is ribbon

Exercise:

$K \# -K$ is ribbon

Question: If K_1 and $K_1 \# K_2$ are ribbon, is K_2 ribbon?

Theorem:

$\boxed{\text{If } K \text{ is slice, then } \sigma(K) = 0}$

Example: $\sigma(RHT) = -2$ so trefoil is not slice

Corollary:

$\sigma : \mathcal{C} \longrightarrow 2\mathbb{Z}$

surjective homomorphism

$\Rightarrow$ RHT infinite order in $\mathcal{C}$

Proposition:

If K is slice, then for any Seifert surface F, there exists a basis for $H_1(F)$ such that the associated Seifert matrix has the form

$$\left(\begin{array}{c|c} B & C \\ \hline D & 0 \end{array}\right)$$

B, C, D square integral matrices

i.e. Seifert form is metabolic, ie vanishes on a half-dim subspace

to prove, we need following:

Lemma

If M is a compact, connected, oriented 3-mfd with $\partial M = \Sigma_g$, then there exists a basis for $H_1(\partial M)$

If M is a compact, connected, oriented 3-mfd with $\partial M = \Sigma_g$, then there exists a basis for $H_1(\partial M)$ represented by 1-cycles, half of which bound rational 2-chains in M

going to do homology with rational coefficients

"half lives, half dies"

proof:

$\mathbb{Q}$ coefficients

$\Rightarrow$ homology groups are vector spaces

$f: V \longrightarrow W$ linear map of vector spaces

rank V = rank$(\ker f)$ + rank$(\operatorname{im} f)$

consider exact sequence of the pair $(M, \partial M)$

$$\begin{array}{ccc} H_3(M,\partial M) & \longrightarrow & H_2(\partial M) \\ \cong & & \cong \\ \mathbb{Q} & & \mathbb{Q} \end{array}$$ is an isomorphism

$$\begin{array}{ccc} H_0(\partial M) & \longrightarrow & H_0(M) \\ \cong & & \cong \\ \mathbb{Q} & & \mathbb{Q} \end{array}$$ is an isomorphism

thus,

$$0 \longrightarrow H_2(M) \longrightarrow H_2(M,\partial M) \longrightarrow H_1(\partial M) \longrightarrow H_1(M) \longrightarrow H_1(M,\partial M) \longrightarrow 0$$

even dimensional

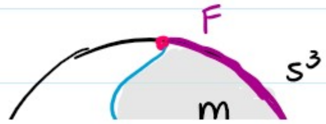
Poincaré Lefschetz duality:

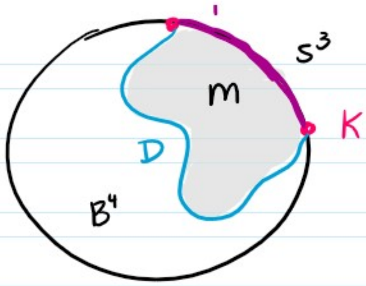
$$H_1(M,\partial M) = H^2(M) = H_2(M)$$

$$H_2(M,\partial M) = H_1(M)$$

Exercise: finish proof

Proof of Proposition:





K is slice
F seifert surface

If K bounds a slice disk D and a Seifert surface F, then there exists $M^3 \subset B^4$ such that

$$M \cap S^3 = F$$

and $\partial M = F \cup D$

Exercise:

$$H_1(F) \cong H_1(F \cup D) = H_1(\partial M)$$

Lemma $\Rightarrow$ $\exists$ basis $\{x_i\}_{i=1}^{2g}$ for $H_1(F)$ such that $x_{g+1}, \dots, x_{2g}$ bound rational 2-chains in M

Exercise:

Check that $lk(x_i, x_j^+) = 0$ for $g+1 \leq i, j \leq 2g$

hint: Rolfsen ch. 5 section D exercise 9

$lk(J,K) =$ signed count of points in $A \cap B$ where A, B are 2-chains in B^4 s.t. $\partial A = J$, $\partial B = K$

Exercise: (linear algebra)

Proposition $\Rightarrow$ $\sigma(K) = 0$ if K is slice

Theorem (Fox-Milnor)

up to a factor of $\pm t^n$

$$\boxed{\text{If } K \text{ is slice, then } \Delta_K(t) \doteq p(t)p(t^{-1}) \text{ for some } p(t) \in \mathbb{Z}[t]}$$

proof:

$$S = \left(\begin{array}{c|c} B & C \\ \hline D & 0 \end{array}\right)$$

$\underset{2g \times 2g}{S}$ matrix

$$\Delta_K(t) = \det(S - tS^T)$$

$$= \det\left(\begin{array}{c|c} B - tB^T & C - tD^T \\ \hline D - tC^T & 0 \end{array}\right)$$

$$= \det(D - tC^T)\cdot \det(C - tD^T)$$

$$= (-t^g)\det(C - tD^T)\det(C - t^{-1}D^T)$$

///

Corollary:

$\boxed{\text{If } K \text{ is slice, then } |\Delta_K(-1)| \text{ is a perfect square.}}$

Example:

K 

figure 8

$\sigma(K) = 0$

$\Delta_K(t) = t - 3 + t^{-1}$

$|\Delta_K(-1)| = 5$ not slice.

Exercise:

 = 

K $\qquad$ $-K$

$K = -K$ and K is not slice

$\Longrightarrow K$ is order 2 in $\mathcal{C}$

(negative amphichiral)

$K \rightsquigarrow S$ Seifert form (up to equivalence)

$-K \rightsquigarrow -S$

$K_1 \# K_2 \rightsquigarrow S_1 \oplus S_2$

K is slice $\rightsquigarrow$ S is metabolic

Theorem: (J. Levine)

$$\boxed{\mathcal{C} \longrightarrow \left(\{\text{Seifert forms}\} \big/ \text{metabolic forms}\ ,\ \oplus \right)}$$

$$\cong \mathbb{Z}^\infty \oplus \mathbb{Z}/_2{}^\infty \oplus \mathbb{Z}/_4{}^\infty$$

signatures $\qquad$ Fox-Milnor condition

defn: Algebraic concordance group $\mathcal{A}$

$\boxed{\text{Corollary:}}$

$\boxed{\mathcal{C} \text{ infinitely generated}}$

$\boxed{\text{Open Question:}}$

Does $\mathcal{C}$ have any torsion beside 2-torsion?

Remark: Can define concordance $\mathcal{C}_n$ in any dimension knotted S^n in S^{n+2}

Kervaine (1965): $\mathcal{C}_n = 0$ for n even

Levine (1969): $\mathcal{C}_n \cong \mathbb{Z}^\infty \oplus \mathbb{Z}/_2{}^\infty \oplus \mathbb{Z}/_4{}^\infty$ n odd, $n \geq 3$

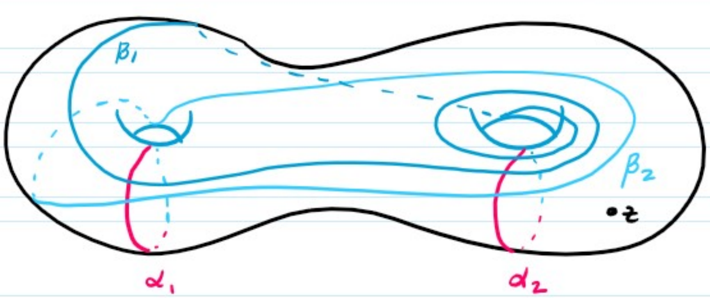
Casson-Gordan (1975): kernel $\mathcal{C} \longrightarrow \mathcal{A}$ is non trivial

Heegaard Floer Homology

- defined by Ozsváth-Szabó
- Lecture Notes on Arxiv on HF homology (by Hom)

3 mfd Y described by a (pointed) Heegaard diagram

fix a base point



α_1, α_2 are in standard position (and in red)

but then β_1, β_2 are determined (an in blue)

$$\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, z)$$

Osváth - Szabó build a chain complex $CF(\mathcal{H})$

- chain homotopy type of $CF(\mathcal{H})$ is an invariant of Y

proof of idea:

Show that the chain homotopy type of $CF(\mathcal{H})$ is invariant under Heegaard moves

isotopies, handleslides, de/stabilizations

$$HF(Y) := H_*(CF(\mathcal{H}))$$

$HF^-(Y)$, $\widehat{HF}(Y)$ depending on which coefficients you use

$\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ (no worry of orientation)

coefficients over polynomial ring $\mathbb{F}[U]$

generators of $CF(\mathcal{H})$: g = genus of surface Σ

(unordered) g-tuples of intersection points between α- and β-circles

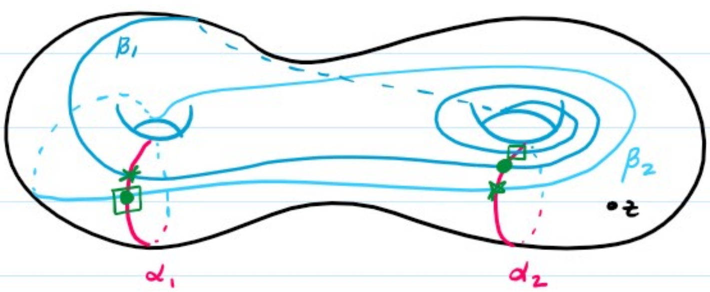
$$\mathrm{Sym}^g\Sigma = \Sigma^g / S_g$$

symmetric group on g elements

$$\mathbb{T}_\alpha = \alpha_1 \times \alpha_2 \times \cdots \times \alpha_g$$

$$\mathbb{T}_\beta = \beta_1 \times \beta_2 \times \cdots \times \beta_g$$

i.e. points in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ are sitting in $\mathrm{Sym}^g\Sigma$



Example: this pair of points is generator a ●

second generator b □

third generator c ★

(3 total generator choices)

↓ as seen here

	β_1	β_2
α_1	1	1
α_2	2	1

← table of # of intersections

for each permutation

$1(1) + 2(1) = 3$ generators total

Chain complex

$$\widehat{CF}(\mathcal{H}) = \langle a, b, c \rangle_{\mathbb{F}}$$

$$CF^-(\mathcal{H}) = \langle a, b, c \rangle_{\mathbb{F}[U]}$$

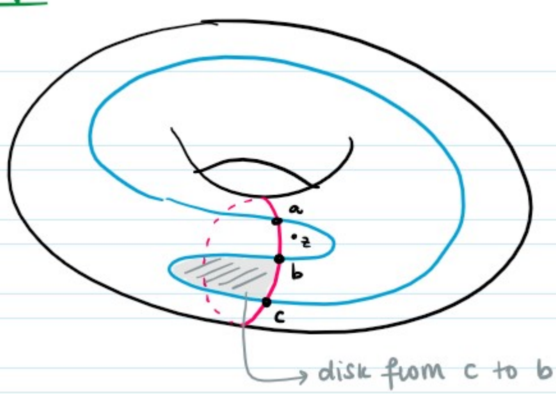
degree $U = -2$

differential: counts holomorphic disks in $\mathrm{Sym}^g \Sigma$

$$x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$$

$$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi) = 1}} \# \widehat{\mathcal{M}}(\phi) U^{n_z(\phi)} y$$

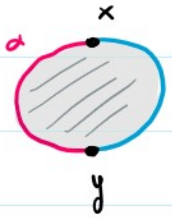
Example:



Roughly, the idea for the differential is to look for disks

↳ disk from c to b

Roughly, the idea for the differential is to look for disks like this



(disk from x to y)

and U counts how many times you see base point in a disk

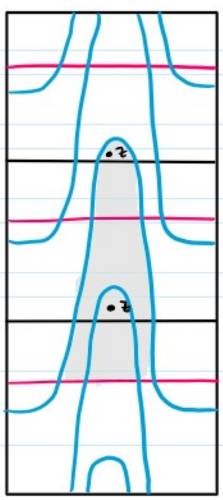
$\therefore$ $\partial c = b$

$\partial b = 0$ (no disk from b)

$\partial a = Ub$

$\left.\begin{array}{l} gr(Ub) = gr(a) - 1 \\ \quad \| \\ gr(b) - 2 \\ gr(b) = gr(c) - 1 \end{array}\right\}$

Q: How could $\bullet z$ show up in multiple disks?



← sufficiently complicated enough and it happens

$$CF^-(\mathcal{H}) = \langle a, b, c \rangle_{\mathbb{F}[U]}$$

$$HF^-(Y) = H_*(CF^-(\mathcal{H})) = \ker \partial / \operatorname{Im} \partial$$

$$= \langle b, a + Uc \rangle_{\mathbb{F}[U]} / \langle b \rangle_{\mathbb{F}[U]}$$

U equivariant so Ub is

u equivariant so ub is also in here

$$= \langle a + uc \rangle_{\mathbb{F}[u]}$$

$$\cong \mathbb{F}[u]$$

weaker invariant

$$\widehat{CF}(\mathcal{H}) = \langle a, b, c \rangle_{\mathbb{F}} \qquad (\text{set } u = 0)$$

$\partial a = 0$

$\partial b = 0$

$\partial c = b$

equivalently, don't allow any disks that cross the basepoint

$$\widehat{HF}(Y) = H_*(\widehat{CF}(\mathcal{H})) = \langle a, b \rangle_{\mathbb{F}} / \langle b \rangle_{\mathbb{F}}$$

$$= \langle a \rangle_{\mathbb{F}}$$

$$\cong \mathbb{F}$$

There is a short exact sequence of chain complexes

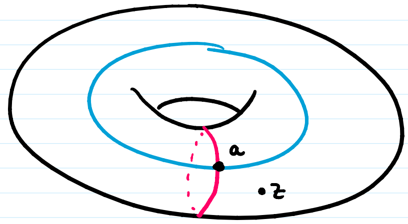
$$0 \longrightarrow CF^-(\mathcal{H}) \xrightarrow{\cdot u} CF^-(\mathcal{H}) \xrightarrow{\text{set } u=0} \widehat{CF}(\mathcal{H}) \longrightarrow 0$$

gives a long exact sequence on homology

$$\begin{tikzcd} HF^-(Y) \arrow[rr, "\cdot u"] & & HF^-(Y) \arrow[dl] \\ & \widehat{HF}(Y) \arrow[ul] & \end{tikzcd}$$

is an exact triangle

Example:



$\partial a = 0$

$$\widehat{HF}(Y) = \langle a \rangle_{\mathbb{F}} \cong \mathbb{F}$$

$$HF^-(Y) = \langle a \rangle_{\mathbb{F}[u]} \cong \mathbb{F}[u]$$

Example: Poincaré Homology Sphere

$$\widehat{HF}(PHS) = \mathbb{F}$$

$$HF^-(PHS) = \mathbb{F}_{(2)}[u]$$

$\downarrow$
$gr\, 1 = 2$

Example: S^3

$$\widehat{HF}(S^3) = \mathbb{F}$$

$$HF^-(S^3) = \mathbb{F}_{(0)}[u]$$

grading distinguishes these

Example:

$$\widehat{HF}\left(\Sigma(2,3,7)\right) = \mathbb{F}^3$$

$$HF^-\left(\Sigma(2,3,7)\right) = \mathbb{F}[u] \oplus \mathbb{F}$$

Exercise: deduce this using exact triangle once you know that $HF^-(\Sigma(2,3,7)) = \mathbb{F}[u] \oplus \mathbb{F}$

Example:

$$\widehat{HF}\left(L(p,q)\right) = \mathbb{F}^p$$

$$HF^-\left(L(p,q)\right) = \left(\mathbb{F}[u]\right)^p$$

Proposition:

If Y is a $\mathbb{Q}HS^3$, $\dim\left(\widehat{HF}(Y)\right) \geq \left|H_1(Y;\mathbb{Z})\right|$

sketch:

$\exists\ \mathbb{Z}/2\mathbb{Z}$ grading on $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$

grading of intersection pt = sign of intersection point

matrix of intersections in $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$ similar to matrix for H_1

definition:

A $\mathbb{Q}HS^3$ Y is an L-space if $\dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$

"Heegaard Floer homology lens space"

Knot Floer Homology

Ozsváth-Szabó, J. Rasmussen

doubly pointed Heegaard diagram

$\mathcal{H} = (\Sigma, \vec{\alpha}, \vec{\beta}, w, z)$

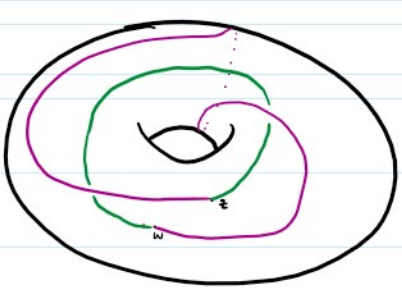
if you want your knot in S^3, you want $\Sigma, \vec{\alpha}, \vec{\beta}$ to be a Heegaard diagram for S^3

Example:

Claim: z,w specify a knot

- Find an arc connecting z,w and misses every β curve — "outside"
- Find an arc connecting z,w and misses every α curve — on "inside"

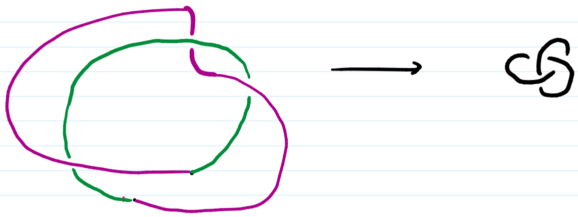
to determine crossing



the isotopy class of knot is uniquely specified

(only 1 way to connect z,w)

knot



Knot Floer complex:

$CFK(\mathcal{H})$ chain complex over $\mathbb{F}[u,v]$

$$\partial a = ub$$
$$\partial b = 0$$
$$\partial c = vb$$

$$CFK^-(\mathcal{H}) = CFK(\mathcal{H}) / _{v=0}$$

$$HFK^-(K) := H_*(CFK^-(\mathcal{H}))$$

Now working w/ a P.I.D

$$\partial a = ub$$
$$\partial b = 0$$
$$\partial c = 0$$

$$= \langle b, c\rangle_{\mathbb{F}[u]} / \langle ub \rangle_{\mathbb{F}[u]}$$

$$\cong \mathbb{F}[u] / \mathbb{F}$$

$$\widehat{CFK}(\mathcal{H}) = CFK(\mathcal{H}) / _{u=v=0}$$

$$\partial a = 0$$
$$\partial b = 0$$
$$\partial c = 0$$

$$\widehat{HFK}(K) := H_*(\widehat{CFK}(\mathcal{H}))$$

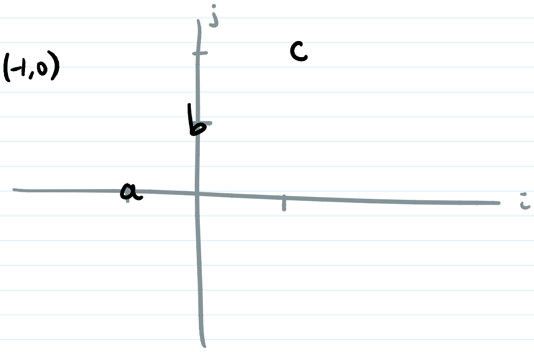
$$= \langle a, b, c \rangle_{\mathbb{F}}$$

$$= \mathbb{F}^3$$

$$\widehat{HFK}(K) = \bigoplus_{i,j} \widehat{HFK}_j(K,i)$$

Complex is bigraded (see Arxiv notes)

$gr(a) = (-1,0)$



Theorem: Ozsváth-Szabó

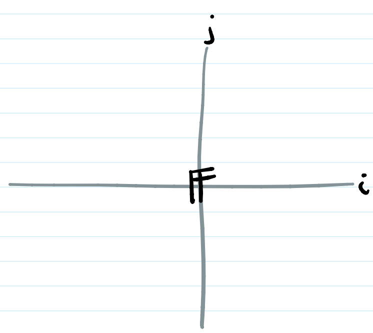
$\widehat{HFK}(K)$ categorifies $\Delta_K(t)$

$$\Delta_K(t) = \sum_{i,j} (-1)^j t^i \dim \widehat{HFK}_j(K,i)$$

$$\text{So } \Delta_K(RHT) = (-1)^0 t^{-1} + (-1)^1 t^0 + (-1)^2 t^1$$
$$= t^{-1} - 1 + t$$

Example:

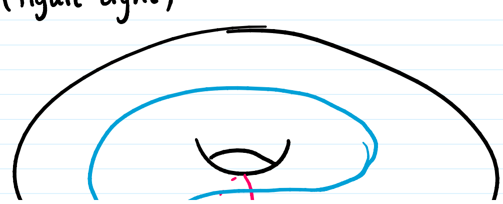
$\widehat{HFK}$(unknot)



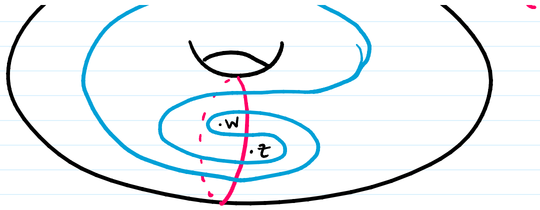
$\Delta_u(t) = 0$

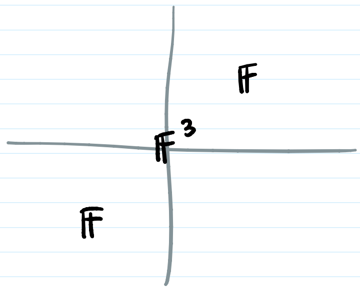
Example:

$\widehat{HFK}$(figure eight)



(exercise: check diagram)

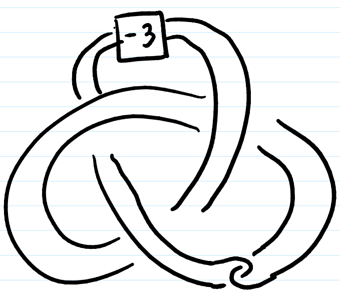


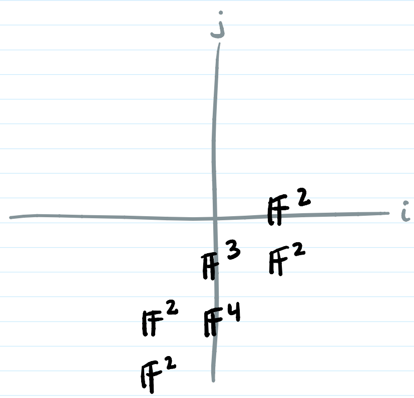


$\Delta_K(t) = -t^{-1} + 3 - t$

Example due to Hedden with interesting Euler char

$\widehat{HFK}\left(Wh^+(RHT)\right)$





$\Delta_K(t) = 1$