Week 2, 8/29 and 8/31

Tuesday, August 29, 2023 10:47 AM

Last time: lens spaces

(see Rolfsen Ch. 9.B for more)

handles

(see Gompf & Stipsicz Ch. 4.1–4.5 for more)

Let X be an n-mfd with boundary and $0 \le k \le n$

An **n-dimension k-handle** is $D^k \times D^{n-k}$ attached to the boundary ∂X along $(\partial D^k) \times D^{n-k}$ via an embedding $f: (\partial D^k) \times D^{n-k} \longrightarrow \partial X$

$X = B^3$

3-dim 1-handle $D^1 \times D^2$

attached along $\underbrace{(\partial D^1) \times D^2}_{\{0,1\} \times D^2}$

What happened to boundary?

B^3 boundary S^2 became T^2

$$\partial(D^k \times D^{n-k}) = (\partial D^k) \times D^{n-k} \amalg D^k \times (\partial D^{n-k})$$
$$= S^{k-1} \times D^{n-k} \amalg D^k \times S^{n-k-1}$$

Given a 3-mfd M whose boundary is T^2, a **Dehn filling of M** is

$$Y = M \cup_f D^2 \times S^1$$
$$f: T^2 \longrightarrow T^2$$

Y is determined by a s.c.c. $\gamma \in \partial M$ that is the image of a meridian $\mu = S^1 \times \{pt\} \subset D^2 \times S^1$

Recall: attaching a solid torus to M can be done in two steps:

(1) First attach $D^2 \times D^1$ along $\partial D^2 \times D^1 = S^1 \times D^1$

$\downarrow$

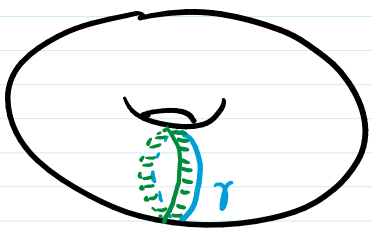
specified by a non-null homologous in ∂M s.c.c. γ

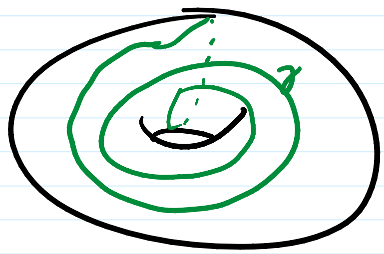
this implies that the resulting ∂ is S^2

(exercise: Check this)

⑪ Attach a 3-ball along resulting S^2 boundary

$\partial M = T^2$





attach thickened disk

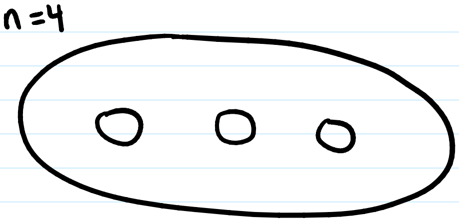
resulting boundary is S^2

SEIFERT MANIFOLDS (generalizations of lens spaces)

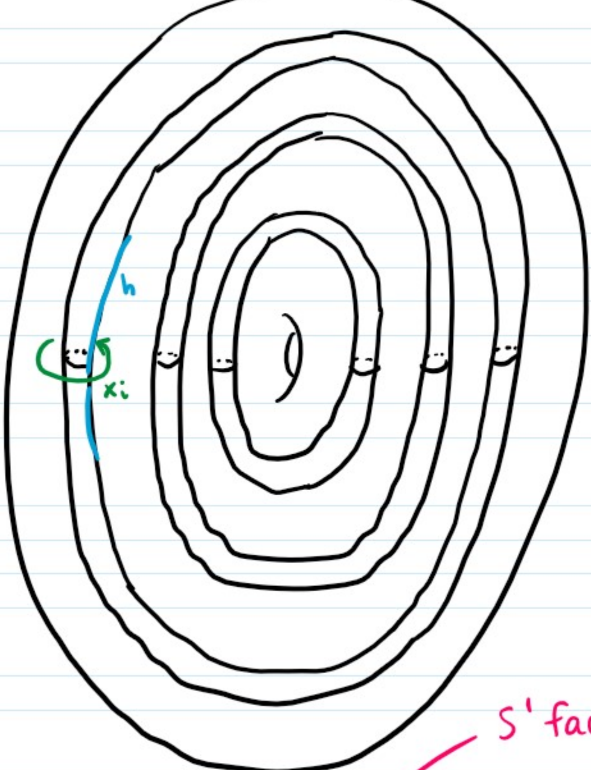
$F = S^2 \setminus \text{int}(D_1^2 \cup \dots \cup D_n^2)$

genus zero (can increase genus & do same thing)

$n=4$



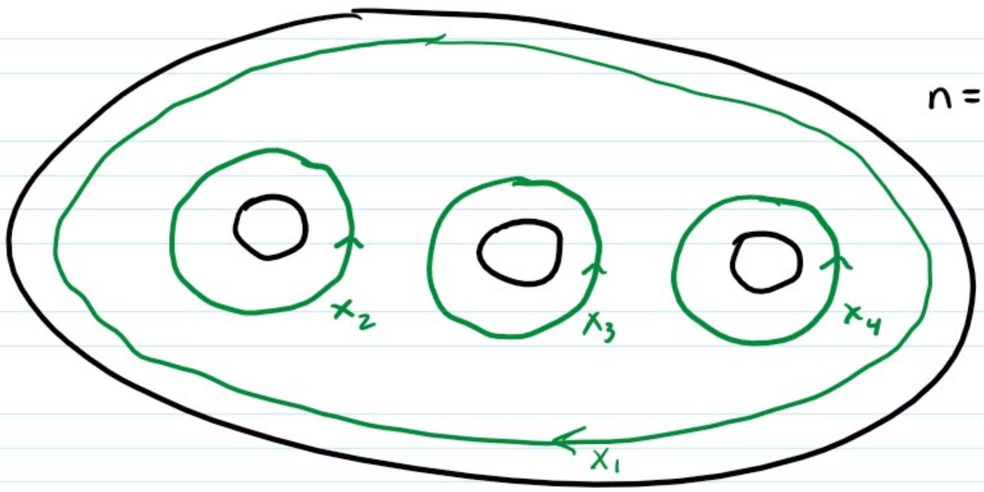
$F \times S^1$



S^1 factor

$$\pi_1(F' \times S^1) = \langle x_1, x_2, \ldots, x_n, h \mid x_i h = h x_i,\ \ x_1 x_2 \ldots x_n = 1 \rangle$$

$n = 4$



exercise: do this yourself w/ Seifert van-Kampen

Want: Dehn fill among the holes

(a_i, b_i) $\quad 1 \le i \le n$ $\quad a_i \ge 2$ $\quad a_i, b_i$ relatively prime

- glue in a solid torus to each boundary component such that the meridian of the i^{th} solid torus is isotopic to $a_i x_i + b_i h$

The image of $\{0\} \times S^1 \subset D^2 \times S^1$ is the i^{th} singular fiber (a.k.a exceptional fibers)

The result is a Seifert manifold $M((a_1,b_1),\dots,(a_n,b_n))$ of genus 0 with n singular fibers

Example:

$n=1$ $M((a_1,b_1)) = L(b,a)$ (exercise: check this)

$n=2$ $M((a_1,b_1),(a_2,b_2))$ is also a lens space

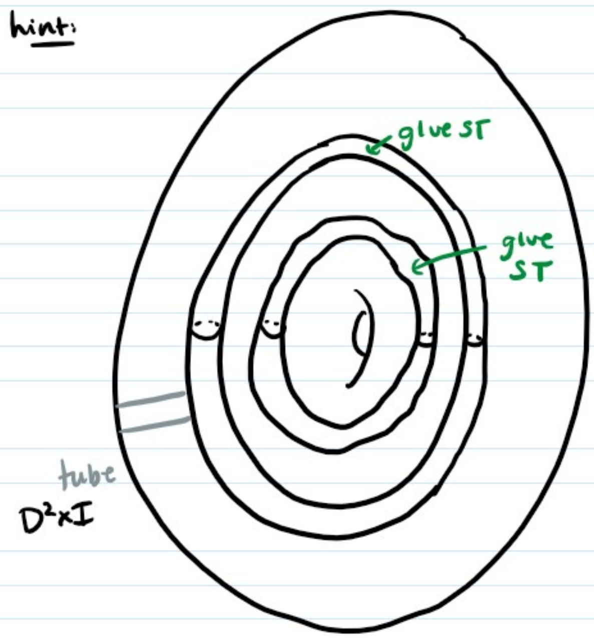
Exercise: show this is a lens space and compute fundamental group (should depend on a_i's and b_i's)

Exercise: Show that a Seifert manifold of genus 0 with at least 3 singular fibers is not a lens space.

hint: Compute π_1 and H_1)

Exercise: Check that a Seifert manifold $M((a_1,b_1),(a_2,b_2),(a_3,b_3))$ has Heegaard genus 2.

hint:



glue an outside solid torus

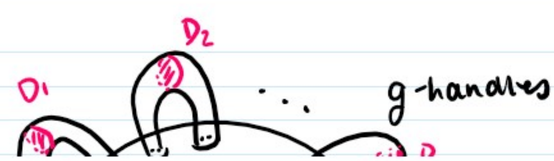
want: two H_2

$H_2 = ST_1 \cup \text{tube} \cup ST_2$

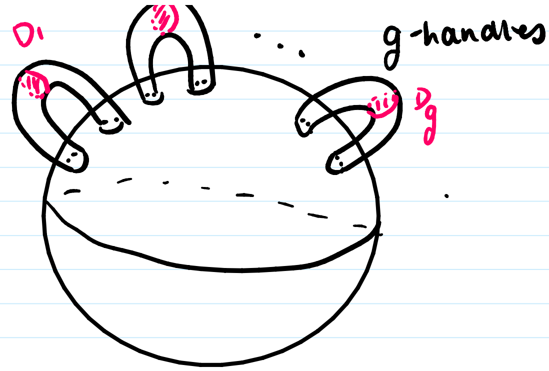
$H_2' = ((D^2 \setminus \text{int}(D^2 \cup D^2) \times S^1) - \text{tube} \cup ST_3$

HEEGAARD DIAGRAMS

$Y = H_g \cup H_g'$ $H_g =$



$Y = H_g \cup H_g'$

$H_g =$ 

Observe: $H_g = B^3 \cup g\ D^2 \times I$

Attach H to H' by first attaching $D_i \times I$ $i=1,\dots,g$ so that the image $\partial D_i \times \{\frac{1}{2}\}$ is α_i

Then attach B^3 ($\exists$ a unique way to do so)

Conclusion: The g-tuple $\alpha_1, \dots, \alpha_g \subset \partial H_g'$ determines Y

Let $\{\alpha_1, \dots, \alpha_g\} \subset \Sigma_g$ is a g-tuple of s.c.c.s that are linearly independent in $H_1(\Sigma_g)$

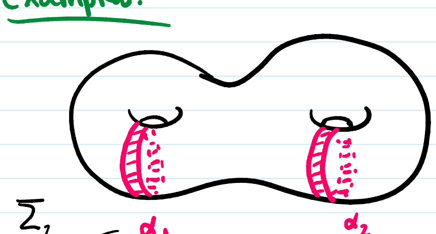
Can attach 2-handles to $\Sigma_g \times [0,1]$ along $\alpha_1 \times \{0\}, \dots, \alpha_g \times \{0\}$

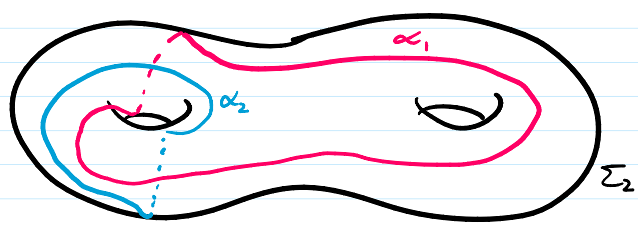
Result is a 3-mfd with two boundary components: Σ_g and S^2

(can see this with an Euler char. argument)

Attach a 3-ball to S^2 boundary and then the result is a genus g handlebody.

Examples:





Exercise: $\alpha_1, \dots, \alpha_g$ are linearly independent in $H_1(\Sigma_g; \mathbb{Z})$

$\Longleftrightarrow$

$\Sigma_g \setminus \alpha_1, \dots, \alpha_g$ is connected

these encode how to fill in the "interior" handlebody

* these encode how to fill in the "interior" handlebody

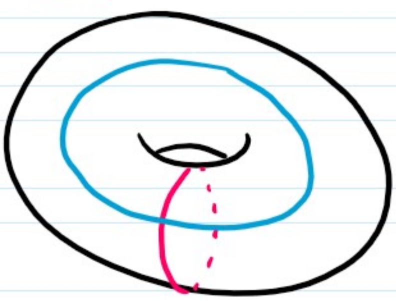
defn: A **Heegaard diagram** is $(\Sigma_g, \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g)$ such that $\alpha_1, \ldots, \alpha_g$ (resp. $\beta_1, \ldots, \beta_g$) are disjoint s.c.c that are linearly independent in $H_1(\Sigma_g)$.

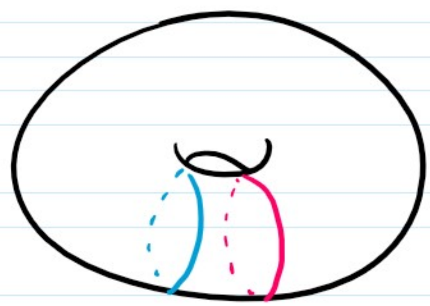
We call the $\alpha_1, \ldots, \alpha_g$ the **attaching curves for H** $= \bar{\alpha}$
and the $\beta_1, \ldots, \beta_g$ the **attaching curves for H'** $= \bar{\beta}$

convention:
$\bar{\alpha}$ Red
$\bar{\beta}$ blue

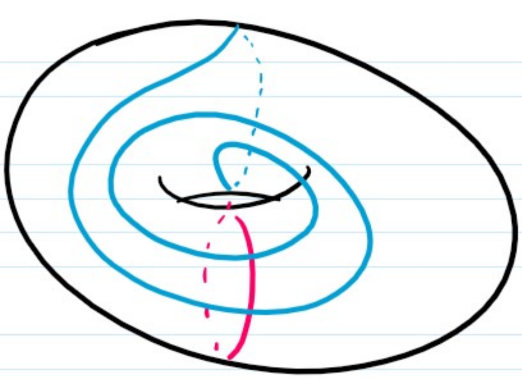
We build a 3-mfd Y from $(\Sigma, \bar{\alpha}, \bar{\beta})$ by taking $\Sigma \times [0,1]$ and attaching 2-handles along $\alpha_i \times \{0\}$ and 2-handles along $\beta_i \times \{1\}$ and capping off the resulting S^2 boundary components with B^3

Example:

1)  $= S^3$

2)  $= S^2 \times S^1$

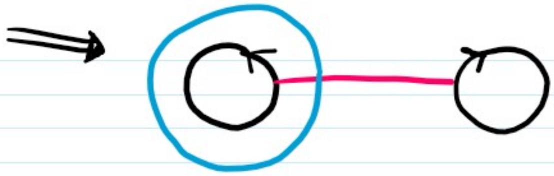
3)  $= S^2 \times S^1$ (stabilization of ex 2)

4)  $= L(2,1)$

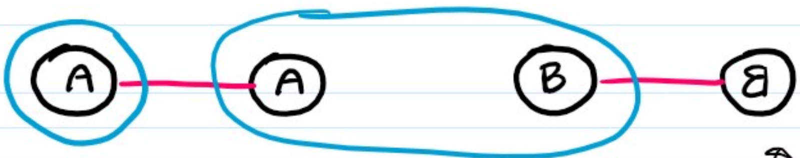
Sometimes we draw Heegaard diagram using the plane $\cup \{\infty\} = S^2$

Example:

1) Identify via a mirror reflection



2)



this is S^3

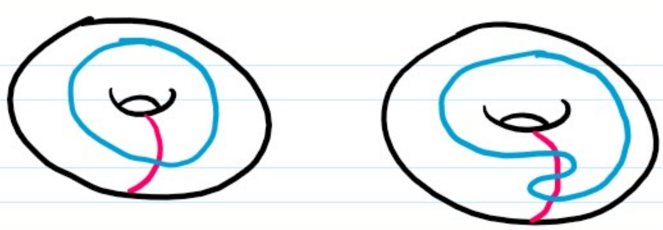
Q: when do two Heegaard diagrams give the same 3-mfd?

Theorem:

$(\Sigma, \vec{\alpha}, \vec{\beta})$ and $(\Sigma', \vec{\alpha}', \vec{\beta}')$ are Heegaard diagrams for the same 3-mfd iff they are related by a finite sequence of the following moves:

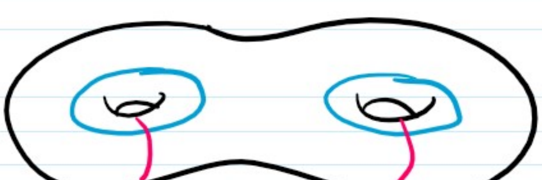
1. isotopies
2. handleslides
3. Stabilization / destabilization

isotopy

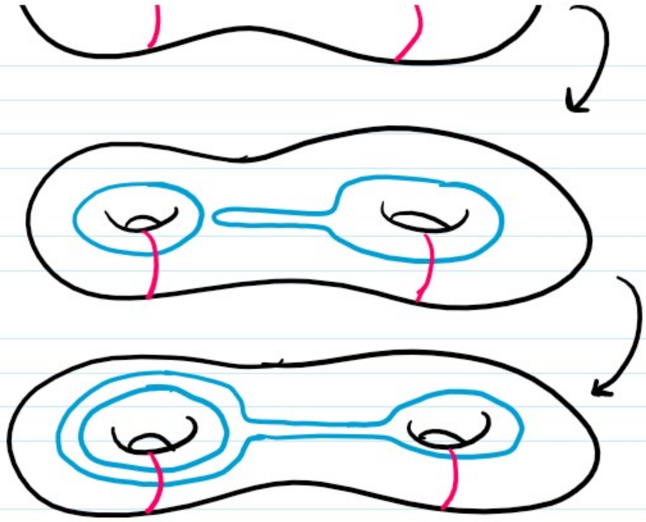


of β-curves
(curves should remain disjoint)
or of α-curves

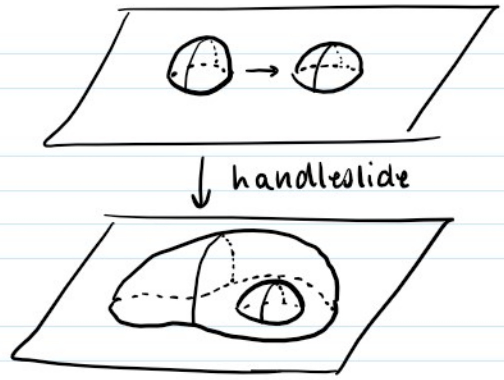
handleslides



involve only α's or only β's



Can think of it as:



Note: isotopies & handleslides do not change genus
(so stabilization will)

Stabilization

connect sum with  equipped with α and β

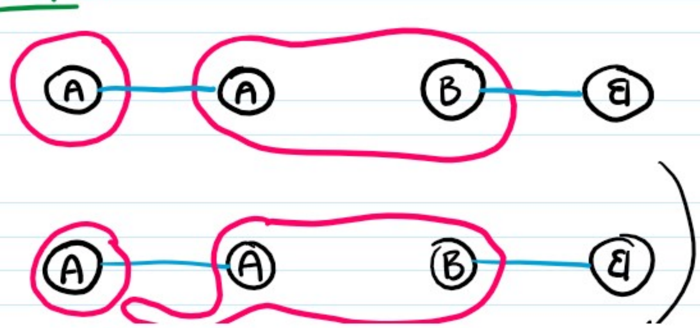
intersecting in a single point

e.g. 

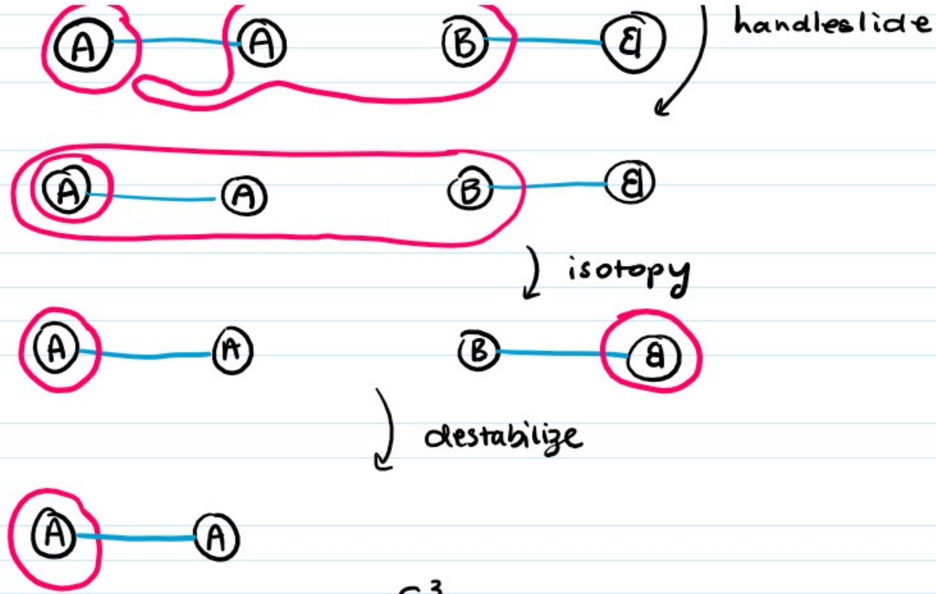
destabilization undoes this operation

For more details & some exercises, see Hom lecture notes on Heegaard Floer homology, section 1 (on arXiv)

Example:

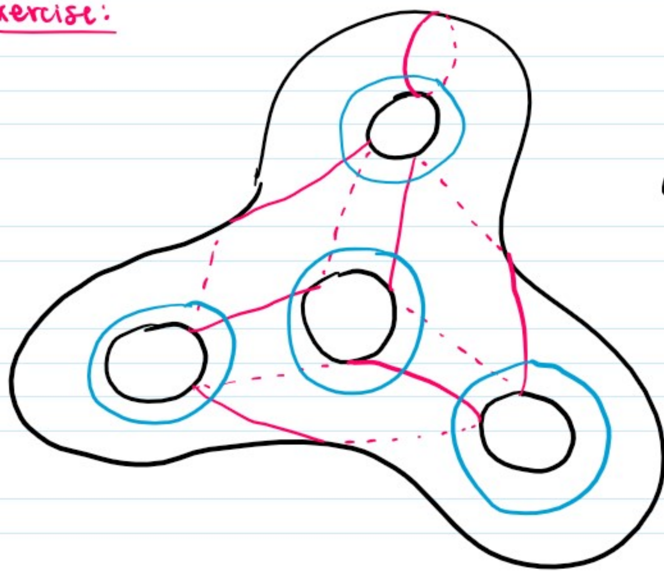


handleslide



this is S^3

Exercise:



what 3-mfd is this?

KNOTS AND LINKS IN 3-MANIFOLDS

A **link** L in Y is a finite collection of smoothly embedded disjoint closed curves

A **knot** is a one component link

Example: in S^3

Example: in S^3

unknot

trefoil
(right-handed)

figure 8

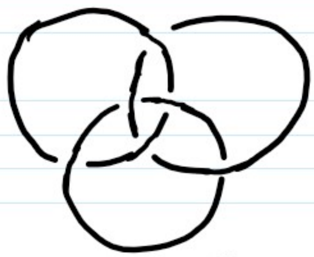
unlink of
two components

Hopf
link

Note:

(left-handed)

Whitehead link



Borromean Rings

Two links L and L' are equivalent if $\exists$ smooth orientation preserving homeo

$$h: Y \longrightarrow Y \quad \text{s.t.}$$
$$h(L) = L'$$

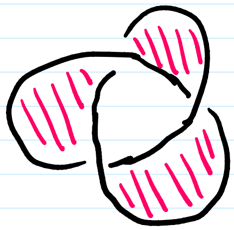
Rmk: Some knots are the same and some are different when we take the mirror

Sometimes our links will be oriented

A Seifert surface for an oriented link $L \subset S^3$ is a compact connected oriented surface $F \subset S^3$ with $\partial F = L$

Example:

Non-Example:



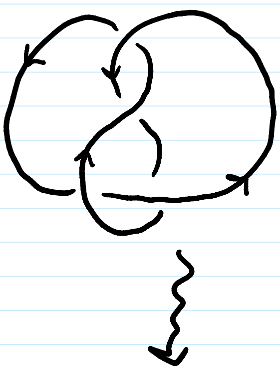
Not orientable
(möbius band)

Theorem:

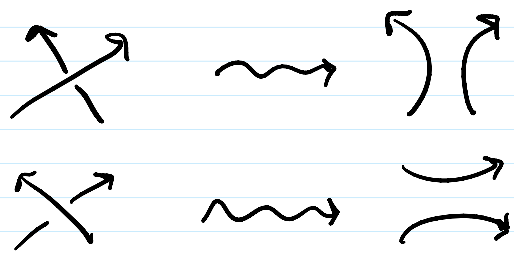
$\boxed{\text{Every link in } S^3 \text{ bounds a Seifert surface}}$

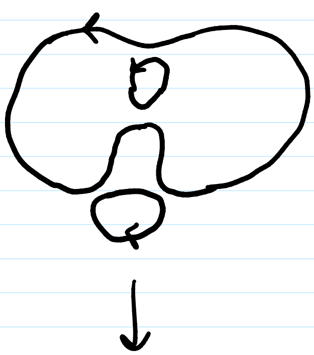
proof:

(uses Seifert's algorithm)



1. Resolve each crossing preserving orientation

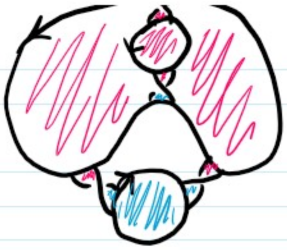




No more crossings in picture
Result is a collection of circles

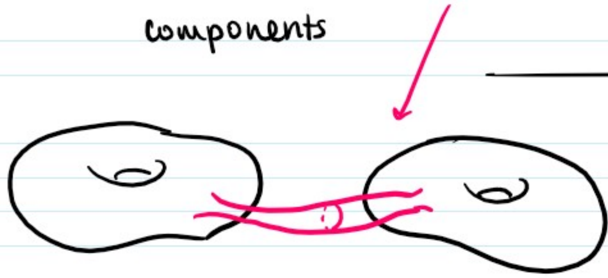
2. Each circle bounds a disk
If they're nested, raise the inner circles

3. Where there used to be crossings, attach bands with twist corresp. to the direction of the crossing



attach bands with twist corresp. to the direction of the crossing

If the resulting surface is connected, we're done. If not, tube together the components



(connect sum)

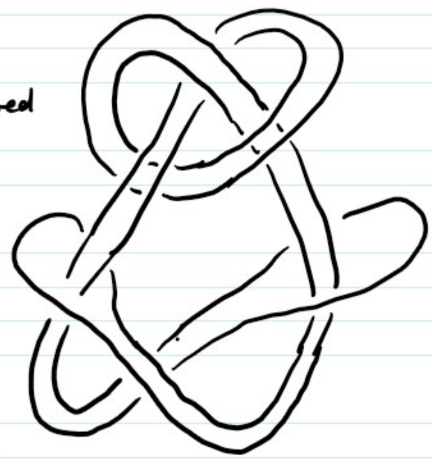
Note: Seifert surfaces are not unique

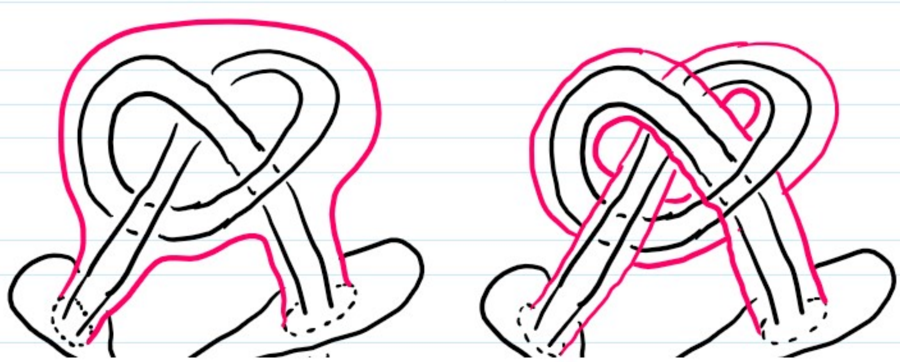
Example:

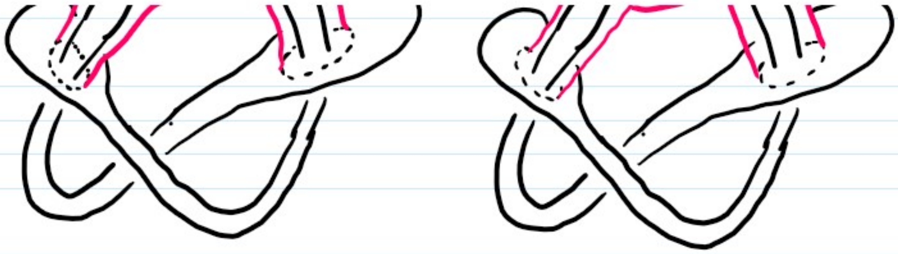
two different Seifert surfaces for unknot

Example:

There is an "obvious" immersed disk that this bounds.







two Seifert surfaces which are homeomorphic
but not equivalent as surfaces embedded in S^3

defn:

the genus of a knot $\subset S^3$ is
$$g(K) = \min\{\text{genus } F \mid F \text{ Seifert surface}\}$$

Fact: K is the unknot $\iff g(K) = 0$

Exercise: Show $H_1(S^3 \setminus K) \cong \mathbb{Z}$
(use long exact sequence of a pair)

this is generated by a meridian