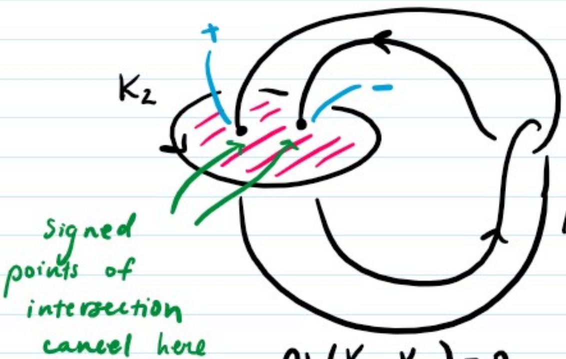
Week 3, 9/5 and 9/7

Tuesday, September 5, 2023 12:31 PM

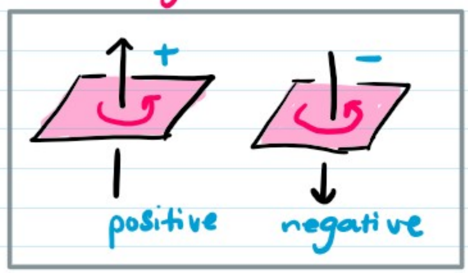
Let K_1, K_2 be disjoint oriented knots in S^3. Their linking number is any of the following equivalent defn's:

① signed count of the number of times K_1 intersects a Seifert surface for K_2



$lk(K_1, K_2) = 0$

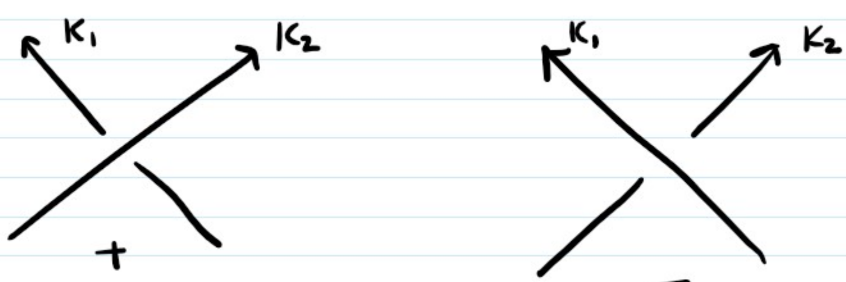
Rmk: it doesn't depend on which Seifert surface you choose



Note: choice of K_1 and K_2 doesn't matter, so linking # is symmetric

$lk(K_1, K_2) = 1$

② signed count of the # of times K_1 crosses under K_2



(or signed count of all crossings between K_1
and K_2 divided by 2)

3 $[K_1]$ is a 1-cycle in $H_1(S^3 - K_2) = \mathbb{Z} = \langle m \rangle$

m=meridian of K

$[K_1] = lk(K_1, K_2) \cdot m$

↳ this def'n makes it clear that this def'n is well-defined

Exercise: Show the linking # is symmetric

ie. $lk(K_1, K_2) = lk(K_2, K_1)$

SURGERY

Note: Dr. Hom is not a medical doctor

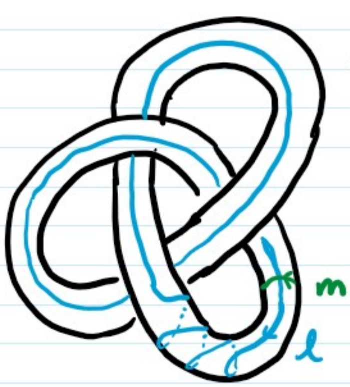
$L \subset Y^3$ remove a tubular nbhd of L and Dehn fill

Let's focus on $K \subset S^3$

$X_K = S^3 - \overline{\nu(K)}$ is called the exterior of K

Recall: $H_1(X_K) = \mathbb{Z}$

X_K:



preferred basis for $H_1(\partial X_K) = \mathbb{Z}^2$

m = meridian

l = longitude

longitude ℓ s.t. $[\ell] = 0 \in H_1(X_K)$

$\Updownarrow$

$$\ell k(\ell, K) = 0$$

$$\Updownarrow$$

$$\ell = F \cap \partial X_K, \ F \text{ Seifert Surface for } K$$

ℓ is called the canonical longitude when $[\ell] = 0$

Rmk: up to orientation, m and ℓ are unique

Any s.c.c. $\gamma \subset \partial X_K$ is isotopic to $p \cdot m + q \cdot \ell$

for $p, q \in \mathbb{Z}$, p, q relatively prime

We often write $\frac{p}{q}$

note: $\frac{1}{0}$ corresponds to m and gives back S^3

$\quad \overset{\shortparallel}{\text{“}\infty\text{ filling”}}$

$$S^3_{p/q}(K) = \text{Dehn filling of } X_K \text{ along } \gamma = p \cdot m + q \cdot \ell$$

$\uparrow$

surgery coefficient

example: $S^3_\infty(K) = S^3$

If $p/q \in \mathbb{Z}$ (ie. $q = \pm 1$), this is called integral surgery

otherwise it is called rational surgery

Note:

1) In general, for $K \subset Y$ there is not a canonical longitude.

(there is a canonical longitude if Y is an $\mathbb{Z}HS^3$)

2) The notion of integral surgery still makes sense for $K \subset Y$

↳ γ runs exactly once along some longitude

Theorem (Lickorish and Wallace)

Every closed orientable 3-mfd Y can be obtained by surgery (integral surgery) on a link in S^3.

do surgery along each component of the link

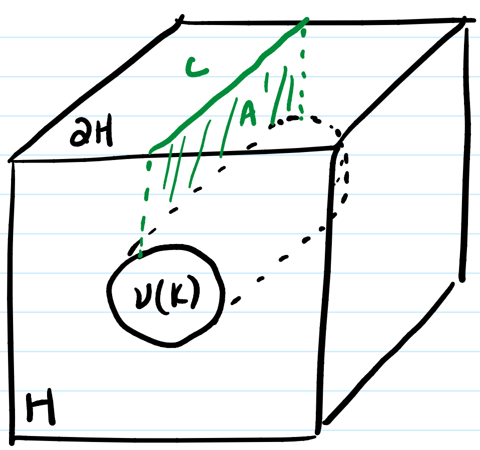
Lemma

Let $h_1, h_2 : \partial H \longrightarrow \partial H'$ be homeomorphs s.t. $h_1 = h_2 \circ \tau_c$ where τ_c is a Dehn twist along a s.c.c. $c \subset \partial H$. Then $Y_2 = H \cup_{h_1} H'$ obtained from $Y_1 = H \cup_{h_1} H'$ by an integral surgery along a knot $K \subset Y_1$ that is isotopic to c

proof of lemma:

Push c into H to obtain knot $K \subset H$

A = annulus connecting c and $\partial \overline{\nu(K)}$



$\varphi : H - \nu(K) \longrightarrow H - \nu(K)$

- cut along annulus A
- twist one end by 360°

· reglue

<u>Note:</u> $\varphi|_{\partial H} = \tau_c$

$\varphi|_{\partial \overline{\nu(K)}}$ = twist along longitude $A \cap \overline{\nu(K)}$

Let $Y_i' = (H - \nu(K)) \cup_{h_i} H'$ $\quad i = 1,2$

$\Phi : Y_1' \longrightarrow Y_2'$ homeomorphism

$$\Phi(x) = \begin{cases} \varphi(x) & x \in H - \overline{\nu(K)} \\ x & x \in H' \end{cases}$$

these agree for x on boundary because $h_1 = h_2 \circ \tau_c$

$$\begin{tikzcd} \partial H \arrow[r, "h_1"] \arrow[d, "\tau_c"'] & \partial H' \arrow[d, "id."] \\ \partial H \arrow[r, "h_2"] & \partial H' \end{tikzcd}$$

$Y_1 - \nu(K)$ homeom. to $Y_2 - \nu(K)$

$\Rightarrow$ Y_2 is obtained from Y_1 by surgery along K

Surgery is integral since Φ maps a meridian m to $m \pm \ell$

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<u>proof of theorem:</u>

$Y = H_g \cup_{h_2} H_g'$ for some genus g, h_2

$S^3 = H_g \cup_{h_1} H_g'$ for some g, h_1

convention: orient.'n reversing

so $h_2^{-1} h_1$ is orient'n preserving

$h_2^{-1}h_1$ homeo on surface of genus g, so

$$h_2^{-1}h_1 \in \mathrm{Mod}(\Sigma_g)$$

so $h_2^{-1}h_1 = \tau_{c_1} \cdots \tau_{c_n}$ where τ_{c_i} is a Dehn twist along c_i

By the lemma, composing with a Dehn twist corresponds to integral surgery along a knot, so a sequence of Dehn twists corresponds to surgery on a link.

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Exercise: ① $H_1(S^3_{p/q}(K)) = \mathbb{Z}/p\mathbb{Z}$

② $L(2,1) \# L(2,1)$ is not surgery along any knot in S^3, $S^3_{p/q}(K)$

sol'n:

$$H_1(S^3_{p/q}(K)) = \mathbb{Z}_p$$

$$H_1(L(2,1) \# L(2,1)) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

A link in S^3 together with a reduced fraction p/q for each component is called a framed link

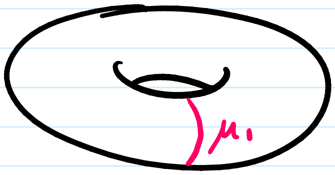
Example:

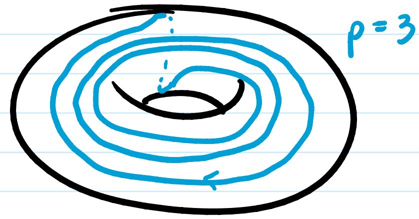
$L(p,1)$ $\quad p \geq 2$

$L(p,1) =$ solid torus (μ_1, λ_1) $\cup_f$ solid torus (μ_2, λ_2)

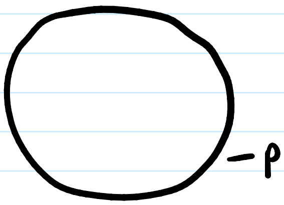
$$f = \begin{pmatrix} -1 & 0 \\ p & 1 \end{pmatrix}$$

μ_1 gets attached to $-\mu_2 + p\lambda_2$



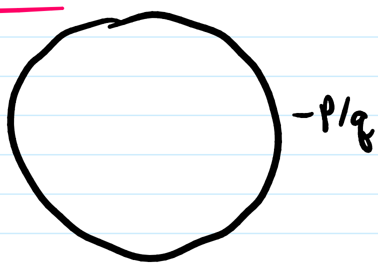


$-\mu_2 + p\lambda_2 = \ell - pm$ in the knot complement (of the unknot)



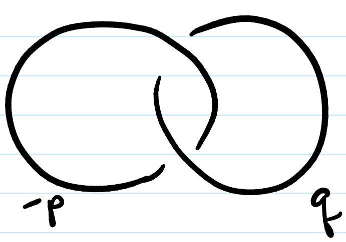
$=$ $(-p)$-surgery

Exercise:

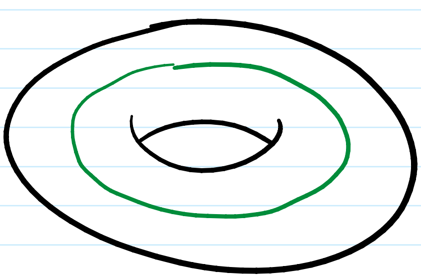


$(-p/q)$-surgery

Example:

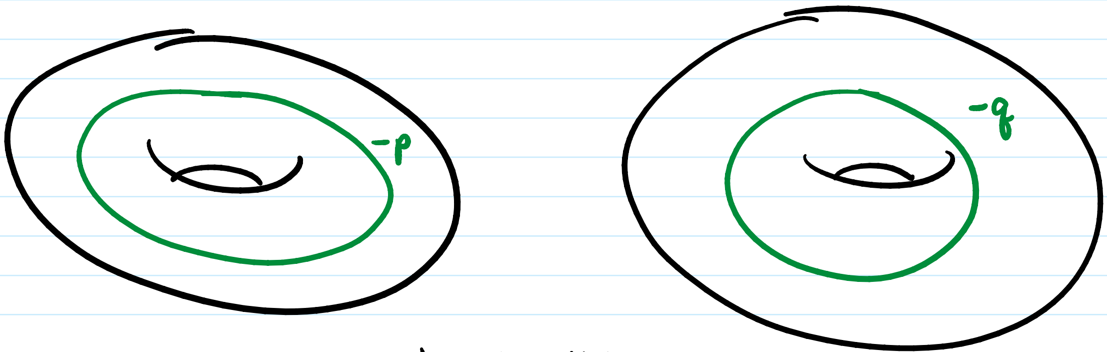


<u>claim:</u> this is $L(pq-1, q)$



Let's do $(-p)$-surgery on the core and $(-q)$-surgery on core of another solid torus

core of solid torus



glue together via $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$\begin{pmatrix} -1 & 0 \\ p & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ q & 1 \end{pmatrix} = \begin{pmatrix} -q & -1 \\ pq-1 & p \end{pmatrix}$$

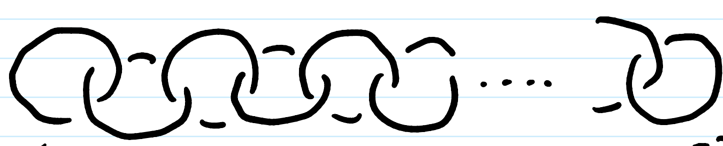
$$= L(pq-1, q)$$

Remarks:

- Complement of Hopf link is $T^2 \times I$
- you cant get everything from $L(pq-1, q)$

Exercise:

Any lens space $L(p,q)$ has an integral surgery description



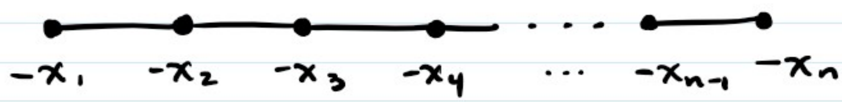
$p/q = [x_1, \dots, x_n]$ is a continued fraction decomposition

$$[x_1, \dots, x_n] = x_1 - \cfrac{1}{x_2 - \cfrac{1}{x_3 - \cfrac{1}{x_4 - \cdots}}}$$

Fact: any fraction p/q can be written in this way

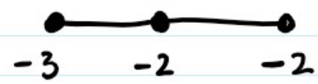
Remarks:

- Sometimes we draw that picture as

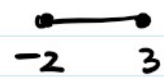


- these are not unique.

Exercise: $L(7,3)$



$$\frac{7}{3} = 3 - \frac{1}{2 - \frac{1}{2}}$$



$$\frac{7}{3} = 2 - \frac{1}{-3}$$

SEIFERT MANIFOLDS, revisited

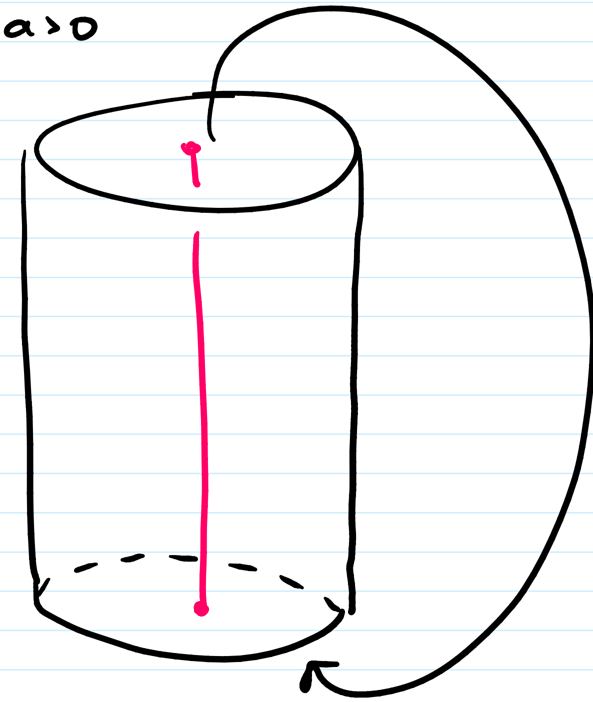
$M((a_1,a_2),\dots,(a_n,b_n))$ Seifert mfds of genus 0 w/ n singular fibers

More generally, a Seifert manifold is a closed 3-mfd together with a decomposition into a disjoint union of circles (called fibers) such that each fiber has a tubular nbhd that forms a standard fibered torus (*)

standard fibered torus:

a, b rel. prime, $a > 0$

$D^2 \times I$



glued with a $\frac{2\pi b}{a}$ twist

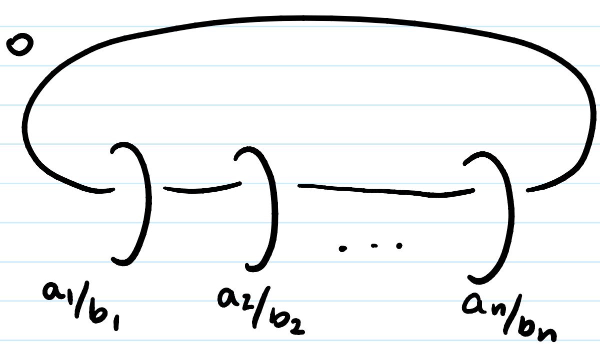
If $a = 1$, then the center is called singular

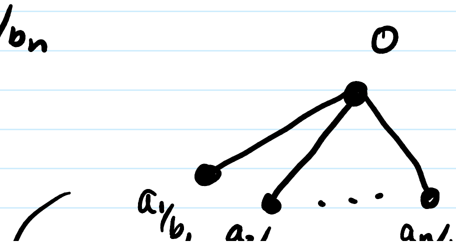
Exercise:

Find a decomposition of $M((a_1,b_1),\dots,(a_n,b_n))$ into a dijoint union of circles satisfies $(*)$

Example:

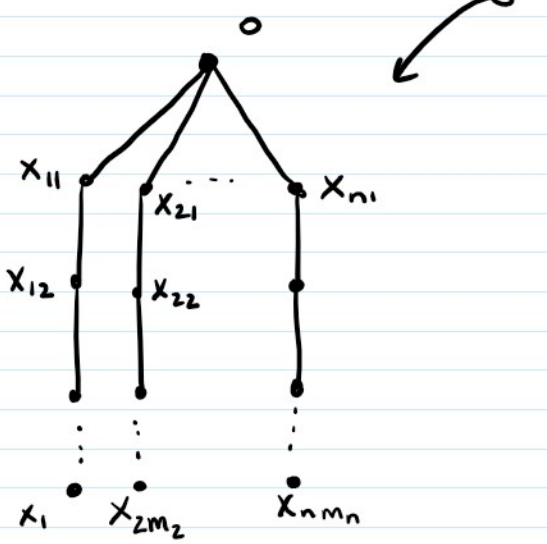
$M((a_1,b_1),\dots,(a_n,b_n))$





$a_i/b_i = [x_{i_1},\dots,x_{i m_i}]$

$a_i/b_i = [x_{i1}, \ldots, x_{im_i}]$



SURGERY AND 4-MFDS

An oriented compact smooth 4-mfd W is called a cobordism between two oriented 3-mfds Y_0 and Y_1 if

$$\partial W = -Y_0 \amalg Y_1$$

If $Y_0 = \emptyset$, then we say that Y_1 is cobordant to zero or null-homotopic

Exercise:

Cobordism is an equivalence relation

Key construction:

An integrally formed link $L \subset Y$ descibes a cobordism:

$K \subset Y$ with a framing $\gamma \subset \partial(Y - \nu(K))$ such that

$$[\gamma] = [K] \in H_1(\nu(K))$$

$W = (Y \times I) \cup_h (\underbrace{D^2 \times D^2}_{\text{2-handle}})$, W called a knot trace

γ determines h in the following way:

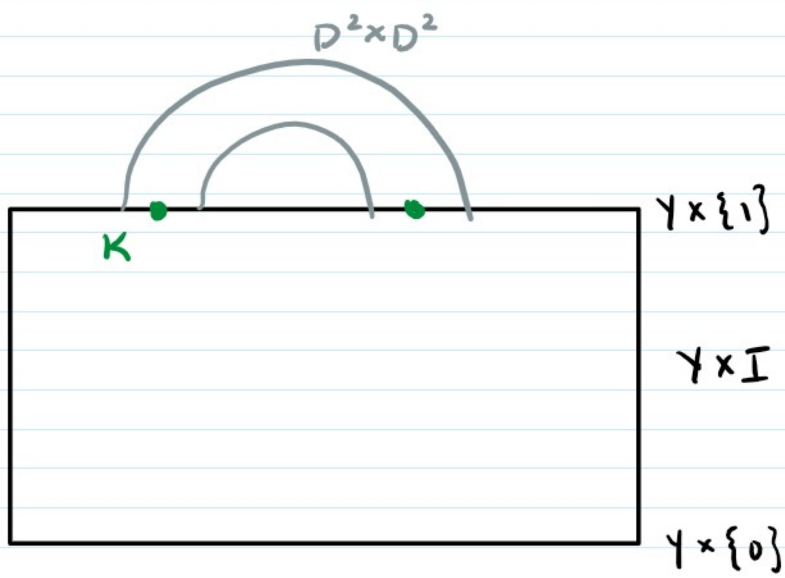
attach a 2-handle $D^2 \times D^2$ to $Y \times \{I\}$

$$h: \underset{\substack{\| \\ S^1}}{\partial D^2} \times D^2 \longrightarrow \overline{\nu(K)} \qquad Y \times \{I\}$$

$$h(S^1 \times \underset{\substack{\| \\ \text{center}}}{\{0\}}) = K$$

$$h(S^1 \times \underset{\substack{\cap \\ \partial D^2}}{\{x\}}) = \gamma$$

Exercise: h is unique up to isotopy.



Theorem: $W = (Y \times I) \cup_h (D^2 \times D^2)$

W is a cobordism between Y and γ-framed surgery on $K \subset Y$

proof: What happened in ∂W?

proof: What happened in ∂W?

1. $Y \times \{0\}$ homeomorphic to Y
2. $Y \times \{1\}$ with $\overline{\nu(K)} = h(\partial D^2 \times D^2)$

   replaced by $D^2 \times \partial D^2$

   meridian $\partial D^2 \times \{x\}$ identified with γ

   ↳ i.e. γ-framed surgery.

Note: W has corners which can be canonically smoothed.

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