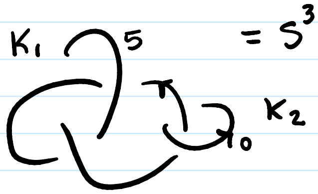
Week 5, 9/19 and 9/21

Tuesday, September 19, 2023 12:31 PM

Example:



$= S^3$

$$\begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$$

the linking matrix

diagonal entries are framings

off-diag. is linking of K_i with K_j

Exercise: linking matrix is a presentation matrix for $H_1(Y)$

Recall: K1 $\mathcal{L} \longrightarrow \mathcal{L} \sqcup O^{\pm 1}$

$$A \longrightarrow \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \pm 1 \end{array}\right)$$

clearly this doesn't change H_1

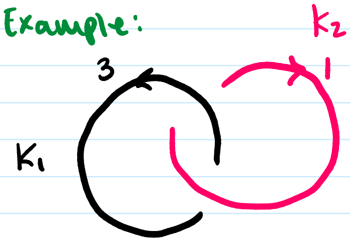
K2: slide K_i over K_j

add (or subtract) j^{th} row to the i^{th} row

and the j^{th} column to the i^{th} column

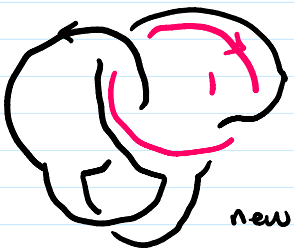
Exercise: check that this ↑ really happens

Example:



$$\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix}$$

$\downarrow$ handleslide K_1 over K_2



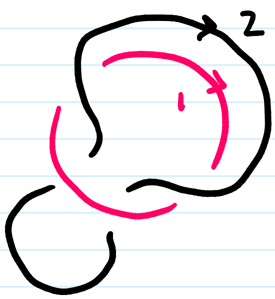
new framing is $3+1+2(lk(K_1,K_2))$

$= 3+1+2(-1)$

$= 2$

$\big\downarrow$ isotope picture

link 0 times



$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

So, $\begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{R_1+R_2} \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix} \xrightarrow{C_1+C_2} \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$

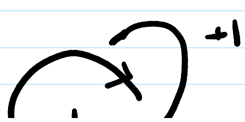
REVERSING ORIENTATION

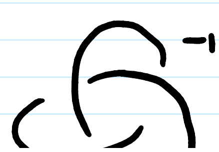
Y surgery on framed link $\mathcal{L}$ in S^3

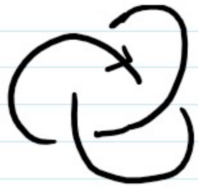
Q: Surgery description for $-Y$?

- reverse orientation on link exterior
- framing n_i becomes $-n_i$

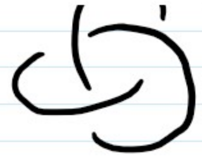
Example:

$Y =$ 

$-Y =$ 

$Y =$ 

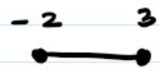
RHT

$-Y =$ 

change all crossing

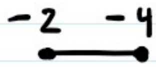
$+1 \longmapsto -1$

Example:

$L(7,3)$ 

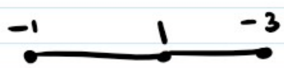
Check continued fraction:

$$\frac{7}{3} = 2 - \frac{1}{-3} = [2,-3]$$

$L(7,4)$ 

$$\frac{7}{4} = 2 - \frac{1}{4} = [2,4]$$

$\downarrow$ same as (after blow-down)



$\downarrow$ blow-down

Recall: $L(p,q) = -L(p,p-q)$

EVEN SURGERIES

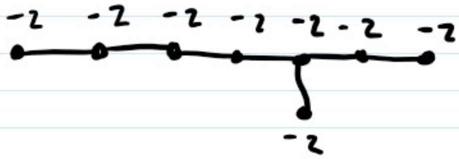
A framed link $\mathcal{L}$ is even if all of it's framings are even.

Theorem:

Any closed oriented 3-mfd is surgery on an even link in S^3

Example:

Poincaré homology sphere



proof uses Kirby moves to eliminate the characteristic sublink

Consider linking matrix A mod 2

↳ doesn't depend on orientation of link

exercise: Check this!

$$A = (a_{ij})$$

$$A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{1,1} \\ a_{2,2} \\ \vdots \\ a_{n,n} \end{pmatrix} \bmod 2$$

Exercise: this system always has a solution

the characteristic sublink is a sublink of $\mathcal{L}$

$$\mathcal{L}' = \{ K_i \subset \mathcal{L} : x_i = 1 \}$$

this always exists and is not unique if $\det(A) = 0 \bmod 2$

Fact: $\mathcal{L}$ is even $\iff$ it has an empty characteristic sublink.

Components of $\mathcal{L}$ are K_i

characteristic sublink is $\mathcal{L}'$

(K_i, x_i) $\quad x_i = 1 \iff K_i \subset \mathcal{L}'$

Example:

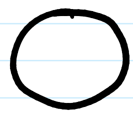
$-3 \quad 1 \quad 5$

$$A = \begin{bmatrix} -3 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 5 \end{bmatrix}$$

$$A \bmod 2 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \leftarrow \text{diagonal entries}$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

So the characteristic sublink is 

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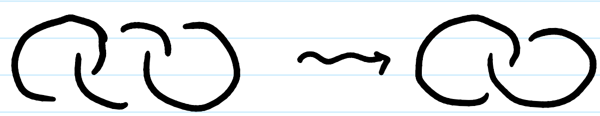
Let's blow-down this component



$-3 \quad 1 \quad 5 \qquad\qquad -4 \quad 4$

Q: How do Kirby moves affect characteristic sublink?

K1: $\mathcal{L} \longrightarrow \mathcal{L} \sqcup O^{\pm 1}$

$\mathcal{L}' \longmapsto \mathcal{L}' \cup (K_{n+1}, 1)$

K2: Slide K_i over K_j

(adds j^{th} row to i^{th} row & j^{th} col to i^{th} col)

$(K_k, x_k) \quad k \neq i, j$ unchanged] Exercise

$$(K_i, x_i) \cup (K_j, x_j) \longmapsto (K_i \# K_j, x_i) \cup (K_j, x_i + x_j)$$

Observation:

If both K_i and K_j were characteristic, then sliding K_i over K_j results in $K_i \# K_j$ characteristic and K_j not

proof of theorem:

$Y =$ integral surgery on $\mathcal{L}$ in S^3

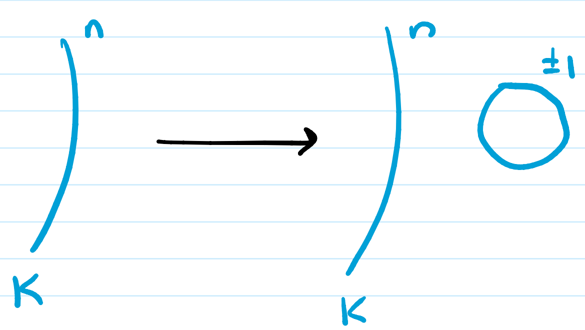
Let $\mathcal{L}'$ be a characteristic sublink.

Use K_2 to reduce the number of components in $\mathcal{L}'$

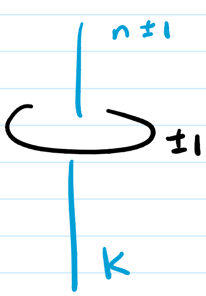
$\Rightarrow$ Can assume $\mathcal{L}'$ consists of a single component K

1.) If K is unknotted (trivial)

blue = in characteristic sublink $\mathcal{L}'$



$\xrightarrow[\text{over } O^{\pm 1}]{\text{slide } K}$



We can change framing on K to ± 1

We can change framing on K to ± 1

then blow down K

Exercise: Check that blow-down does not create any new characteristic components

2.) If K is non-trivial, use Kirby moves to unknot K

See Saveliev

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CLOSED 4-MFDS (connected, oriented)

in dim 4, it matters if your manifold is smooth vs. topological

We will primarily be interested in smooth 4-mfds

Fact: Any finitely presented group can occur as π_1 of a smooth closed 4-mfd

There is no algorithm to tell if 2 finitely presented groups are isomorphic

We will primarily be interested in simply connected 4-mfds.

Interesting invariant: INTERSECTION FORM

X = closed, oriented, connected, simply connected

$$H_4(X) \cong H^0(X) \cong \mathbb{Z}$$

2 generators: $+1, -1$

a choice of generator corresponds to a choice of orientation

Fundamental class $[X]$ = generator for $H_4(X)$

$$\pi_1(X)=0 \Rightarrow H_1(X)=0 \Rightarrow H_3(X)=0$$

P.D.

Lemma

$$\boxed{H_2(X),\ H^2(X) \text{ are both torsion free}}$$

proof:

Universal coefficients theorem.

$$H^2(X) = \mathrm{Hom}(H_2(X),\mathbb{Z}) \oplus \mathrm{Ext}(\underbrace{H_1(X),\mathbb{Z}}_{0})$$

$\Rightarrow H^2$ torsion free.

Poincaré Duality $\Rightarrow H_2(X) = H^2(X)$ also torsion free. //

Define: bilinear form

$$Q_X : H^2(X) \otimes H^2(X) \longrightarrow \mathbb{Z}$$

$$(a,b) \longmapsto \langle a \cup b, [X] \rangle$$

This is called the intersection form of X

- It is symmetric
- Non-degenerate (follows from P.D.)

defn: A form is non-degenerate

$$v \longmapsto (y \longmapsto Q_X(y,v))$$

is an isomorphism

$\Rightarrow Q_X$ as a matrix is invertible over $\mathbb{Z}$

$\therefore$ $\det = \pm 1$ (a unit)

this is called being unimodular

Alternative Perspective:

$$PD: H^2(X) \xrightarrow[\text{isom.}]{} H_2(X)$$

$\alpha \in H_2(X)$ is represented by a smoothly embedded closed oriented surface $F_\alpha \subset X$ if

$$i_*([F_\alpha]) = \alpha \in H_2(X)$$

i = inclusion

$[F_\alpha] \in H_2(X)$ fundamental class of F_α

Abuse notation and write $[F_\alpha]$ for $i_*([F_\alpha])$

Lemma

> X closed oriented smooth 4-mfd
> Any class $\alpha \in H_2(X)$ can be represented by a smoothly embedded closed oriented surface F_α

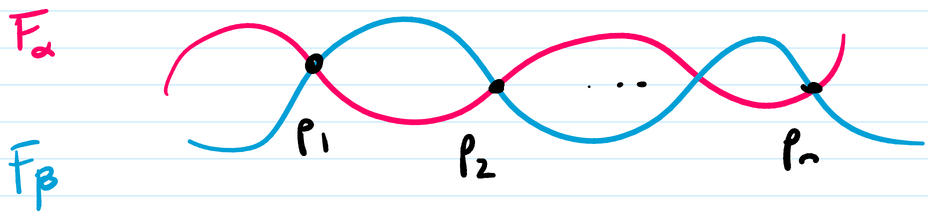
proof: See Saveliev

X Smooth

$\alpha, \beta \in H_2(X)$ represented by F_α and F_β

We can perturb so that F_α and F_β meet transversely

We can perturb so that F_α and F_β meet transversely in finitely many points



p_i = intersection points (signed)

$\varepsilon(p_i)$ = sign of p_i

compare $T_{p_i}F_\alpha \oplus T_{p_i}F_\beta$ to $T_{p_i}X$

$$Q_X' : H_2(X) \otimes H_2(X) \longrightarrow \mathbb{Z}$$
$$(\alpha, \beta) \longmapsto \alpha \cdot \beta = \sum_i \varepsilon(p_i)$$

Lemma:

Q_X' is well-defined and agrees w/ Q_X in the sense that for $a, b \in H^2(X)$ and $\alpha = PD(a)$, $\beta = P.D.(b)$, then

$$\alpha \cdot \beta = \langle a \cup b, [X] \rangle = Q_X(a,b)$$

proof: Bott-Tu section 6

From now on, we will use Q_X to refer to either / both of these forms.

Given a basis $\{e_i\}$ for $H_2(X)$, we will write Q_X as a matrix $e_i \cdot e_j$

as a matrix $e_i \cdot e_j$

Examples:

$$S^2 \times S^2$$

$$H_2(S^2 \times S^2) = \mathbb{Z} \oplus \mathbb{Z}$$

$$\alpha = S^2 \times \{x\} \qquad \beta = \{y\} \times S^2$$

How do these intersect?

$$\alpha \cdot \alpha = 0 \qquad \alpha \cdot \beta = 1$$

$$\beta \cdot \alpha = 1 \qquad \beta \cdot \beta = 0$$

$\Longrightarrow$

Hyperbolic matrix:

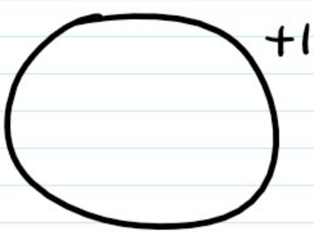
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = H$$

Example:

$$H_2(\mathbb{C}P^2) = \mathbb{Z}$$

generated by $\mathbb{C}P^1$

intersection form $[+1]$

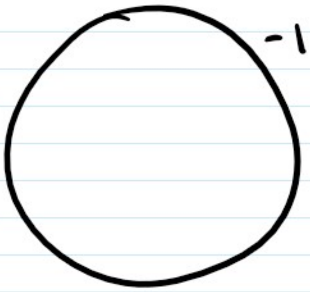


$$\cup_{S^3} D^4 = \mathbb{C}P^2$$

Example:

$$H_2(\overline{\mathbb{C}P^2}) = \mathbb{Z}$$

intersection form $[-1]$



$$\cup_{S^3} D^4 = \overline{\mathbb{C}P^2}$$

Example:

$$H_2(X_1 \# X_2) = H_1(X) \oplus H_2(X)$$

$$Q_{X_1 \# X_2} = Q_{X_1} \oplus Q_{X_2}$$

$$X = p\,\mathbb{C}P^2 \oplus q\,\overline{\mathbb{C}P^2}$$

$$Q_X = \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} = \left[\begin{array}{c|c} \overbrace{\begin{matrix}1 & & \\ & \ddots & \\ & & 1\end{matrix}}^{p\text{-times}} & 0 \\ \hline 0 & \underbrace{\begin{matrix}-1 & & \\ & \ddots & \\ & & -1\end{matrix}}_{q\text{-times}} \end{array}\right]$$

Example:

Kümmer Surface

$$K3 = \left\{ [z_0 : z_1 : z_2 : z_3] \in \mathbb{C}P^3 \;\middle|\; z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0 \right\}$$

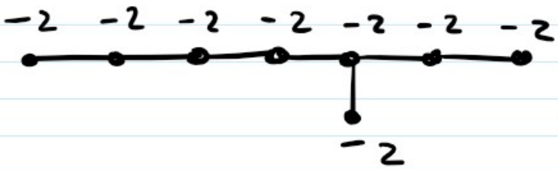
Facts:

1. simply connected
2. closed, oriented 4-mfd
3. intersection form $Q_{K3} = E_8 \oplus E_8 \oplus 3H$

$$E_8 = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & & 0 & \\ 1 & -2 & 1 & 1 & 0 & & & \\ 0 & 1 & -2 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 1 & 0 & 0 \\ & & 0 & 1 & -2 & 1 & 0 & 1 \\ & & & 0 & 1 & -2 & 1 & 0 \\ & 0 & & 0 & 0 & 1 & -2 & 0 \\ & & & 0 & 1 & 0 & 0 & -2 \end{bmatrix}$$

linking matrix of

Recall:



See Saveliev for description of K3

↳ it's a 22-component framed link.

It's a surgery description

↳ attach 2-handles and resulting boundary is S^3, attach B^4 to get K3

Closed 4-mfd $\rightsquigarrow$ unimodular integral intersection form
(entries are integers)

A lattice is a finitely generated free Abelian group

Let $Q: L \otimes L \longrightarrow \mathbb{Z}$ be a unimodular, symm, bilin. form

L = lattice (think of $H_2(X)$)

Example:

intersection form of a closed 4-mfd

3 basic invariants of Q

1. **rank** $\quad \mathrm{rk}\, Q = \mathrm{rk}_{\mathbb{Z}} L = \dim_{\mathbb{R}}(L \otimes \mathbb{R})$

2. **Signature** $\quad \otimes \mathbb{R} \rightsquigarrow$ real symm. bilinear form which can be diagonalized (over $\mathbb{R}$)

ie. $\exists$ a basis $\{e_i\}$ s.t.

$$Q = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$b_+ = \#\text{positive } \lambda_i\text{'s}$$

$$b_- = \#\text{ negative } \lambda_i\text{'s}$$

$$\text{signature} \quad \sigma = \sigma(Q) = b_+ - b_-$$

Q is definite if b_+ or b_- is zero.

Q is positive-definite if $b_- = 0$
negative-definite if $b_+ = 0$
indefinte otherwise

Remark: rank & signature are invariants of Q over $\mathbb{R}$

3. **type**

Rmk: type is an invariant of an integral form

Q is even if $Q(x,x) = 0 \pmod 2 \ \forall x \in L$ (L = lattice)

Q is odd otherwise

When are two forms the same?

$$Q_i : L_i \otimes L_i \longrightarrow \mathbb{Z}$$

$Q_1 \cong Q_2$ are isomorphic if there is

$\varphi : L_1 \longrightarrow L_2$ s.t.

$$\begin{tikzcd} L_1 \otimes L_1 \arrow[rr, "\varphi \otimes \varphi"] \arrow[dr, "Q_1"'] & \arrow[d, phantom, "\circ"] & L_2 \otimes L_2 \arrow[dl, "Q_2"] \\ & \mathbb{Z} & \end{tikzcd}$$

commutes.

rank, signature, and type are invariants of the isomorphism class of Q.

rank, signature, and type of 4-mfd X are rank, signature, and type of Q_X.