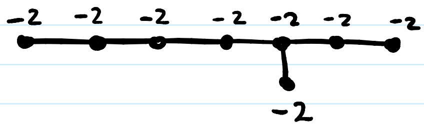
Week 6, 9/26 and 9/28

Tuesday, September 26, 2023 12:32 PM

Two key Examples:

① E_8 = linking matrix of this link:



exercise: check this is even, negative definite, and compute $\sigma(E_8) = -8$

② $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ = linking form on $S^2 \times S^2$

exercise: check it's even and indefinite

check that $\sigma(H) = 0$

Facts about unimodular, symmetric, bilinear integral forms

1.) Q is odd and indefinite

$$\Longrightarrow Q \cong b_+(+1) \oplus b_-(-1)$$

$$\cong \left[\begin{array}{ccc|ccc} +1 & & & & & \\ & \ddots & & & & \\ & & +1 & & & \\ \hline & & & -1 & & \\ & & & & \ddots & \\ & & & & & -1 \end{array}\right] \begin{array}{l} \}\, b_+ \\ \\ \}\, b_- \end{array}$$

$$\operatorname{rank} Q = b_+ + b_-$$

$$\sigma(Q) = b_+ - b_-$$

$$\boxed{b_+ = \frac{rk\,Q + \sigma(Q)}{2} \qquad b_- = \frac{rk\,Q - \sigma(Q)}{2}}$$

2.) Q even $\Longrightarrow \sigma(Q) \equiv 0 \mod 8$

3.) Q even and indefinite

a) If $\sigma(Q) \leq 0$, then $Q \cong a \cdot E_8 \oplus b \cdot H$

a) If $\sigma(Q) \leq 0$, then $Q \cong a \cdot E_8 \oplus b \cdot H$

where $a = \dfrac{-\sigma(Q)}{8} = \dfrac{b_- - b_+}{8}$

$b = \dfrac{rk(Q) + \sigma(Q)}{2} = b_+$

b) If $\sigma(Q) > 0$, then $Q \cong a \cdot E_8 \oplus b \cdot H$

where $a = \dfrac{\sigma(Q)}{8}$

$b = \dfrac{rk Q - \sigma(Q)}{2}$

Upshot:

We understand indefinite forms quite well

Question: How much does the intersection form determine the 4-mfd?

Theorem: (Whitehead)

> If X_1, X_2 are simply connected, closed, oriented 4-mfds, then they are homotopy equivalent iff their intersection forms are isomorphic

Example:

$\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$

odd, $\begin{pmatrix} +1 & 0 \\ 0 & -1 \end{pmatrix}$

$S^2 \times S^2$

even, $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\therefore$ not homotopy equivalent

Theorem (Wall)

Let X_1, X_2 be simply connected, closed, oriented, smooth 4-mfds. If their intersection forms are isomorphic, then $\exists$ $k \geq 0$ s.t.

$$X_1 \# k(S^2 \times S^2) \underset{\text{diffeo}}{\cong} X_2 \# k(S^2 \times S^2)$$

"stably diffeomorphic"

Rmk:

$\exists$ examples where $k \neq 0$ but as far as I know, it is open whether k ever needs to be > 1.

"1 is enough?"

Example: (where $k \neq 0$)

$X_1 = K3 \# \overline{\mathbb{CP}^2}$

$Q_{X_1} = 2E_8 \oplus 3H \oplus (-1)$

$rk(Q_{X_1}) = 16 + 6 + 1 = 23$

$\sigma(Q_{X_1}) = -16 - 1 = -17$

odd (not even)

indefinite

$b_+ = 3$

$b_- = 20$

$\Rightarrow$ odd + indefinite $\Rightarrow$ $Q_{X_1} \cong 3(+1) \oplus 20(-1)$

another 4-mfd w/ Q_{X_1} is $3\mathbb{CP}^2 \# 20\overline{\mathbb{CP}^2} = X_2$

fact: Kronheimer-Mrowka show that X_1 and X_2 are <u>NOT</u> diffeo using Donaldson polynomial

Remark:

$K3 \# \mathbb{CP}^2$ is in fact diffeo to $4\mathbb{CP}^2 \# 19\overline{\mathbb{CP}^2}$
(exercise)

<u>Question:</u> Which forms can be realized as the intersection form of a 4-mfd?

Theorem: (Rokhlin)

If X is a simply connected, closed, smooth, orient. 4-mfd with even intersection form, then

$$\sigma(X) \equiv 0 \pmod{16}$$

Example:

E_8 does not arise as the intersection form of a simply connected, closed, smooth 4-mfd

same with $E_8 \oplus H$

* What if we relaxed <u>smooth</u> condition?

Theorem (Freedman)

Given a unimodular, symmetric, bilinear integral form which is even, then there exists up to homeo exactly one simply conn. closed topological 4-mfd representing that form one simply conn. closed topological 4-mfd representing that form.

If odd, there are exactly two top-4-mfds rep'ing that form.

In the odd case, one of those topological 4-mfds never admits a smooth structure.

Corollary

If X_1, X_2 are closed, simply connected, smooth 4-mfds with $Q_{X_1} \cong Q_{X_2}$ then X_1 and X_2 are homeomorphic

Ex:

$$K3 \# \overline{\mathbb{CP}^2} \simeq_{\text{homeo}} 3\mathbb{CP}^2 \# 20\overline{\mathbb{CP}^2}$$

but

$$K3 \# \overline{\mathbb{CP}^2} \not\cong_{\text{diffeo}} 3\mathbb{CP}^2 \# 20\overline{\mathbb{CP}^2}$$

these are what we call an exotic pair

||

defn: manifolds that are homeo but not diffeo

Ex:

$\exists$ a simply connected, closed topological 4-mfd X with $Q_X = E_8$

X cannot be smooth by Rokhlin's theorem

Corollary

If a topological 4-mfd X is homotopy equivalent to S^4 then it has to be homeomorphic to S^4

"Topological 4-dim Poincaré Conjecture"

Smooth 4-dim Poincaré Conjecture (open)

Every smooth 4-mfd which is homeo to S^4 is in fact diffeo to S^4

Theorem (Donaldson)

If the intersection form is positive definite, then the form is isomorphic over $\mathbb{Z}$ to $(+1)$'s along the diagonal.

" " negative definite,

" " (-1)'s

Remark: the number of positive-definite unimodular symm, unimodular bilinear forms of a fixed rank is finite, but grows rapidly

EX: $>10^{51}$ such forms of rank 40

Donaldson says that only the diagonalizable ones are realized as Q_X for smooth X

Theorem (Furuta)

If the intersection form Q of a smooth, simply conn closed, oriented 4-mfd is even, then

$$\mathrm{rk}\, Q > \frac{10}{8} |\sigma(Q)|$$

Note: If even, indefinite

$Q_X \cong a E_8 \oplus b H$ then

$$\sigma(Q_X) = -8a$$

$$rk(Q_X) = 8a + 2b$$

so Furuta implies $8a + 2b > \frac{10}{8} \cdot 8a$

$$b > a$$

$\boxed{\frac{11}{8}\text{th's conjecture:}}$

> Hypothesis as above:
> $$rk\, Q \geq \frac{11}{8} \left| \sigma(Q) \right|$$

(open)

Ex: K3

$$rk\, Q_{K3} = 22$$

$$\left| \sigma(Q_{K3}) \right| = 16$$

So far, we've described some smooth closed 4-mfds as

$$\underbrace{B^4 \cup n(2\text{-h})\text{'s}}_{\text{resulting boundary} = S^3} \cup B^4$$

(e.g. K3)

But not all 4-mfds admit such a description.

Q: Is there a general way to describe a closed 4-mfd?

answer: yes. as a union of handles

Handle Decomposition:

n-dim k-handle $D^k \times D^{n-k}$

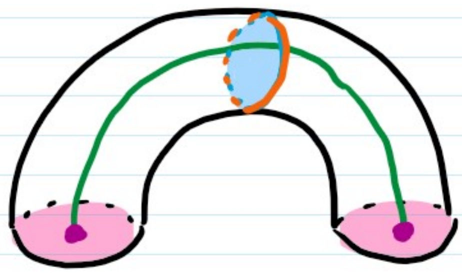
attached along the attaching region $(\partial D^k) \times D^{n-k}$

attaching sphere $(\partial D^k) \times \{0\}$

core $D^k \times \{0\}$

co-core $\{0\} \times D^{n-k}$

belt-sphere $\{0\} \times \partial D^{n-k}$



Exercise: think about what it would look like for other n and k.

Exercise: An integral form Q_X is even

$\iff$

$Q(e_i, e_i) \equiv 0 \mod 2$
for every basis $\{e_i\}$

$\iff$

every matrix representing Q has even diagonal

$\iff$

$Q(e_i, e_i) \equiv 0 \pmod 2$ for at least one basis

$\iff$

$\exists$ at least 1 matrix representation Q

$\Longleftrightarrow$

$\exists$ at least 1 matrix representation Q
where diagonal is even

Example: n-dim 0-handle

$$D^0 \times D^n$$

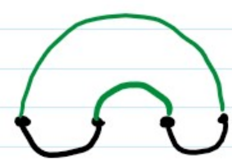
attaching region $= \emptyset$

A handle decomposition of smooth M^n is a description of M by attaching handles

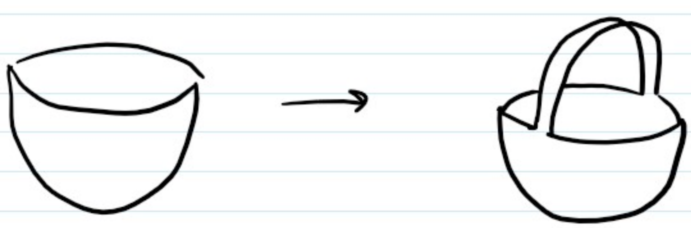
Example:

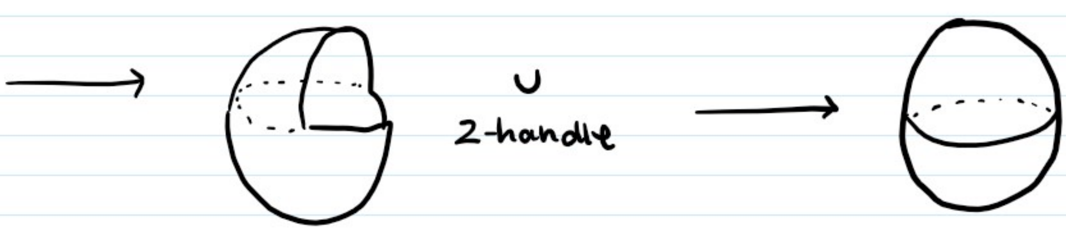
$S^1 =$ 

OR



Example:





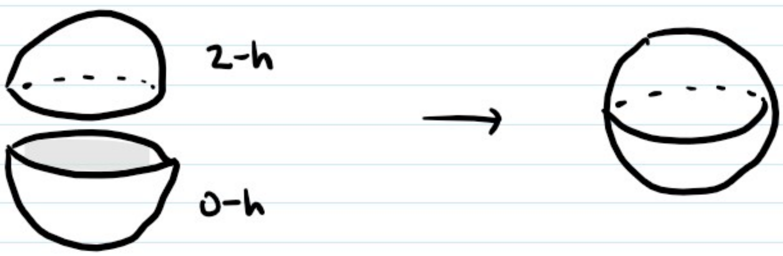
Theorem

$\boxed{\text{Every smooth compact manifold admits a handle decomposition}}$

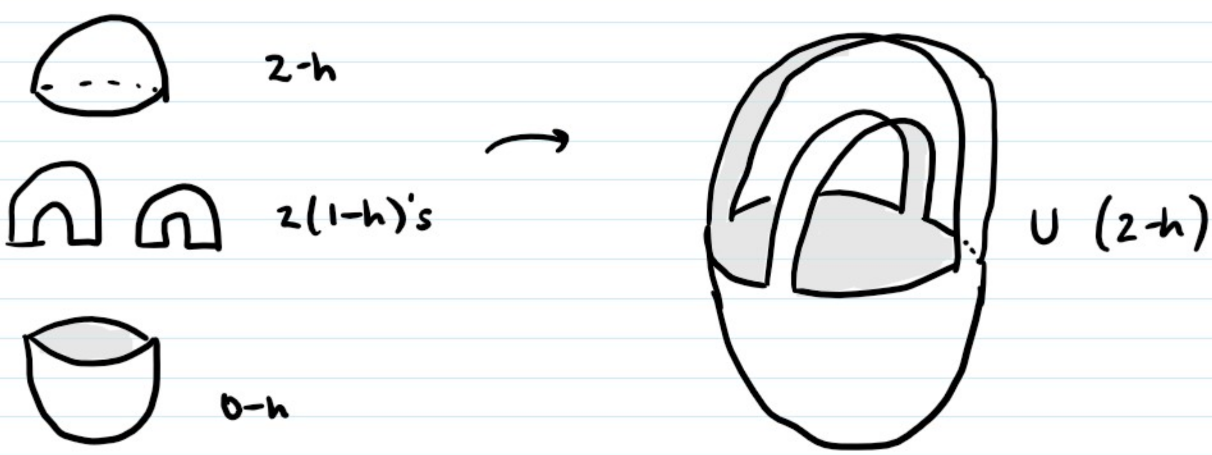
proof: morse theory: critical point of index $k \rightsquigarrow$ k-handle

SURFACES:

S^2

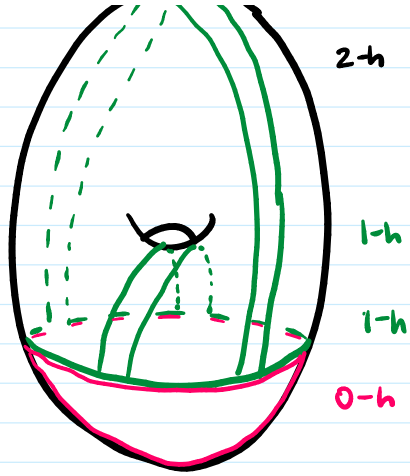


T^2



can visualize:

can visualize.



Exercise: Find a handle decomposition for any closed oriented surface

THREE MANIFOLDS

Heegaard splittings/diagrams

α-handlebody

g (1-h)'s

1 (0-h).

β-handlebody

g (2-h)'s

1 (3-h)

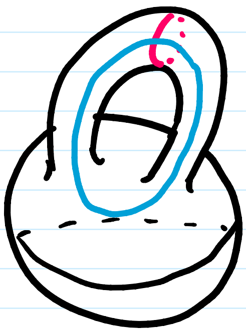
(by flipping factors can think of this as same as left picture)

$$\Sigma = \partial(\alpha\text{-handlebody}) = \partial(\beta\text{-handlebody})$$

α-curves = belt spheres of (1-h)'s
α-disks = co-cores of (1-h)'s

β-curves = attaching sphere of (2-h)'s
β-disks = cores of (2-h)'s

Example:



belt sphere of 1-handle

glue in 2-handle along blue

New boundary is S^2

$\cup$ (3-h)

(attach 3-h to this and gives S^3

Conclusion:
Heegaard diagrams give us handle decompositions.

Proposition

We can always isotope the attaching maps so handles are attached in order of increasing index. Handles of the same index can be attached in any order (or simultaneously.

proof: Gompf & Stipsicz Prop 4.2.7

Proposition

If M^n is compact and connected, then it admits a handle decomp. with exactly 1 (0-h).

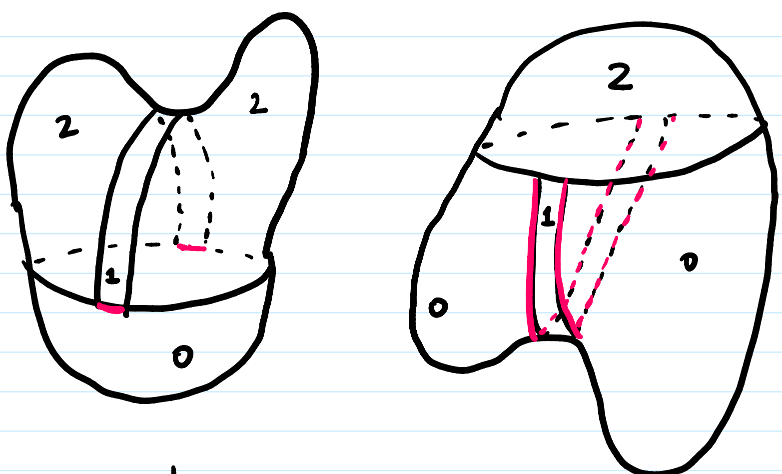
If M^n is compact and connected, then it admits a handle decomp. with exactly 1 (0-h).
If $\partial M \neq \emptyset$, we can also assume there is exactly 1 n-handle

proof: Gompf & Stipsicz Prop 4.2.13

Note:

1) Can turn a handle decomp. "upside down"
n-dim k-handle $\rightsquigarrow$ (n-k)-handle

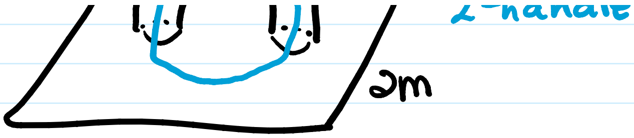
Example:



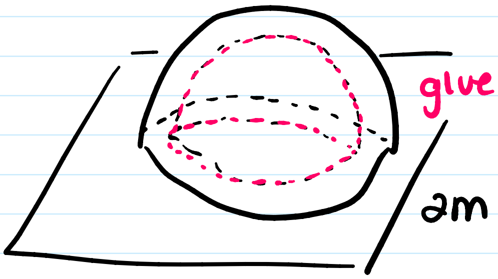
cores and cocores swap
attaching regions in red

2.) Can create "cancelling" pairs of handles
(k-1) handle and k-handle $1 \leq k \leq n$

ex:

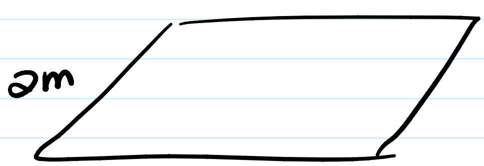


3-dim 2-handle and a 3-handle:



glue in a 3-handle w/ $\partial = S^2$

In general, we can do this:

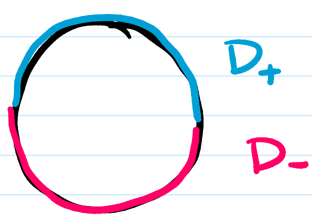


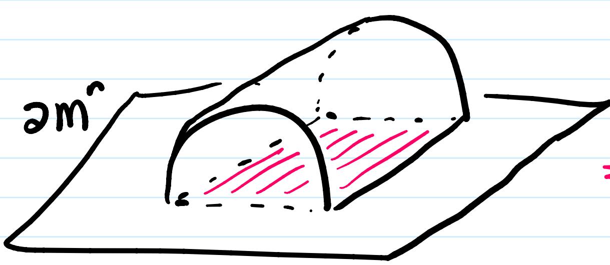
M^n

boundary sum = connect sum the boundaries

$$M^n \natural D^n$$
$$\begin{matrix} \| \\ D^k \times D^{n-k} \end{matrix}$$

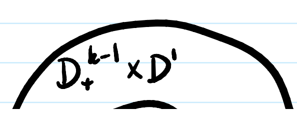
$$\partial D^k = S^{k-1} = D_+^{k-1} \cup D_-^{k-1}$$



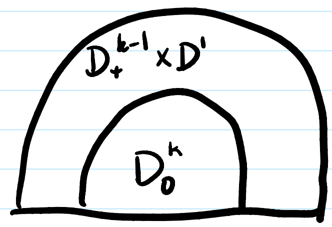


$= D^{k-1} \times D^{n-k}$

M^n



Slice off a nbhd of D_+^{k-1}



$= D^k$

(k-1)-handle is $D_+^{k-1} \times D^1 \times D^{n-k}$ attached to ∂M

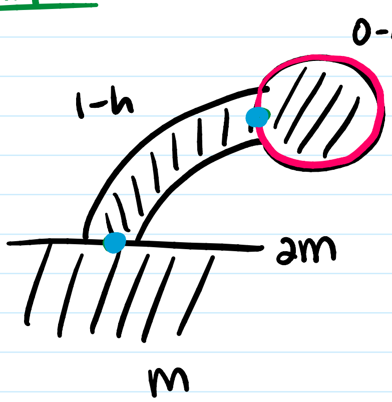
cancelling k-handle $D_0^k \times D^{n-k}$

Proposition

A (k-1) handle h_{k-1} and a k-handle h_k can be cancelled provided the attaching sphere of h_k intersects the belt sphere of h_{k-1} transversely in a single point.

proof: Gompf & Stipsicz Prop 4.2.9

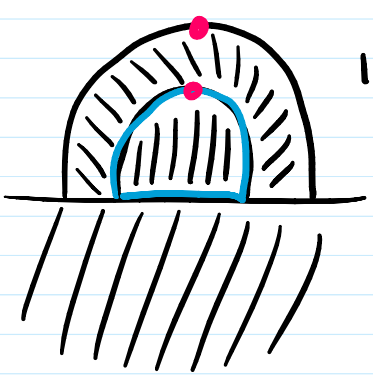
Example:



attaching sphere

belt sphere

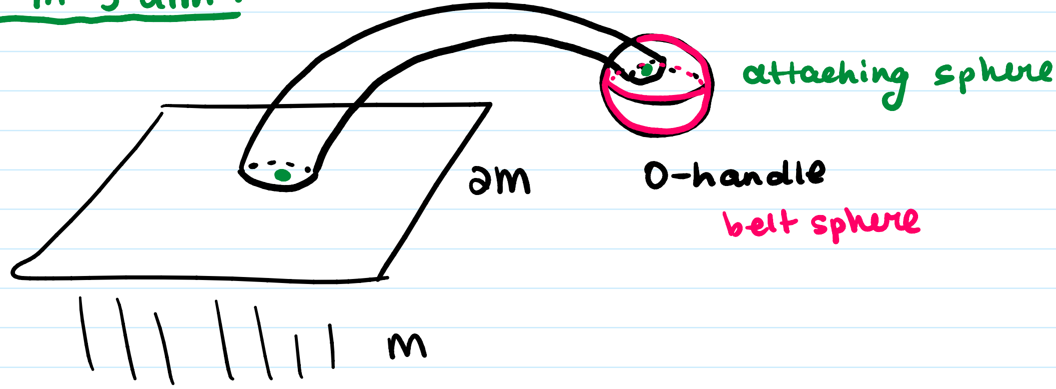
Example:



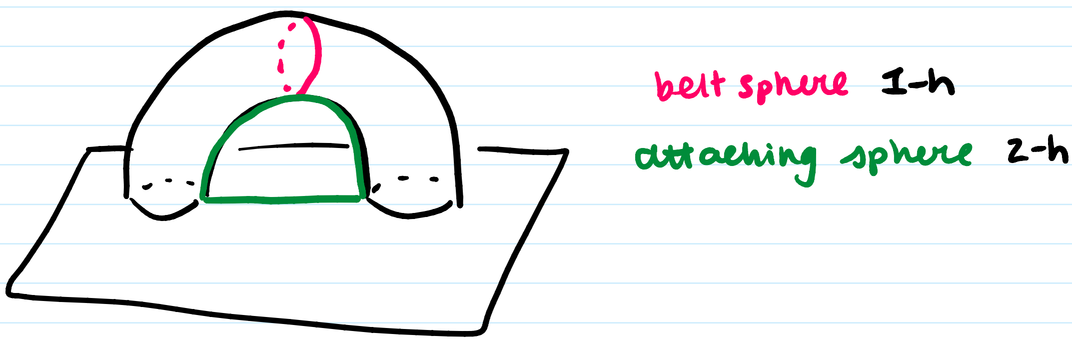
1-handle belt sphere

attaching sphere 2-handle

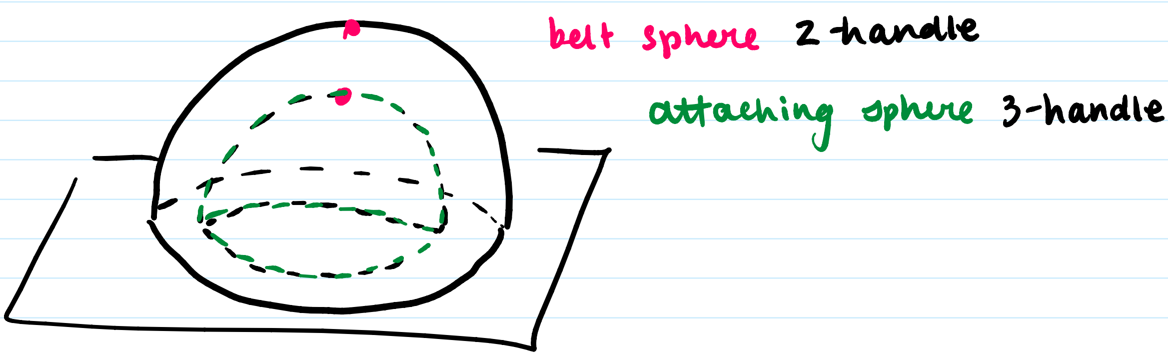
Ex: in 3-dim:



0-, 1- cancelling pair



1-, 2- cancelling pair



2-, 3- cancelling pair

Exercise:

Show that destab of a Heegaard diagram is a 1-, 2- cancelling pair

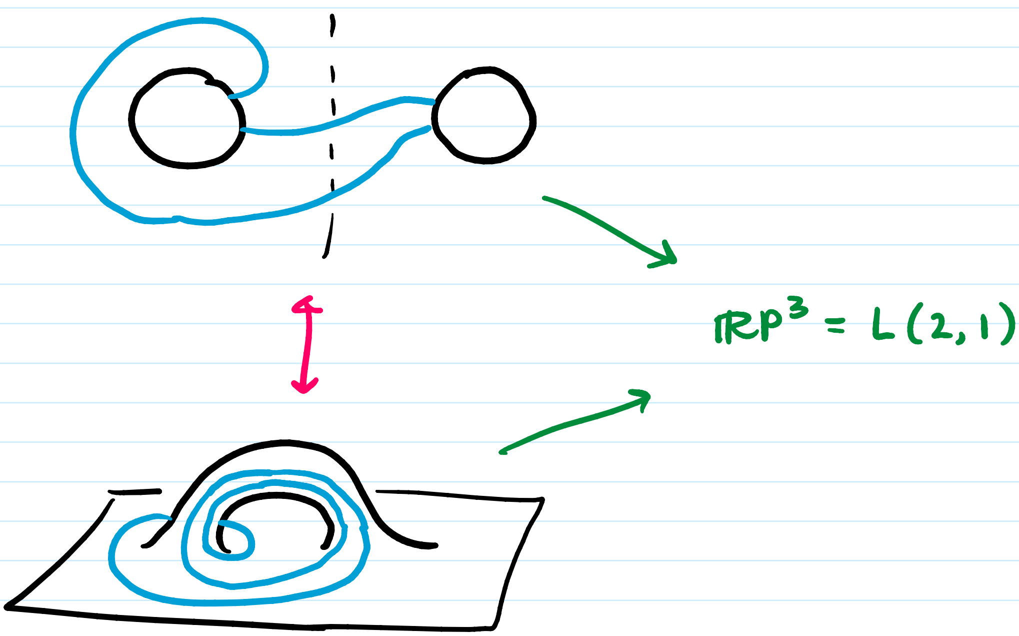
DIMENSION 3: HEEGAARD DIAGRAMS

Unique 0-, 3- handle

$\partial D^3 = S^2 = \mathbb{R}^2 \cup \{\infty\}$ (think of paper as $\mathbb{R}^2$)

attaching region of 1-handles $(\partial D^1) \times D^2 = D^2 \amalg D^2$

attaching spheres of 2-handles $(\partial D^2) \times \{0\}$



If resulting boundary is S^2, attach a 3-handle to get a closed 3-mfd.

to see it's $\mathbb{RP}^3$, look at torus: