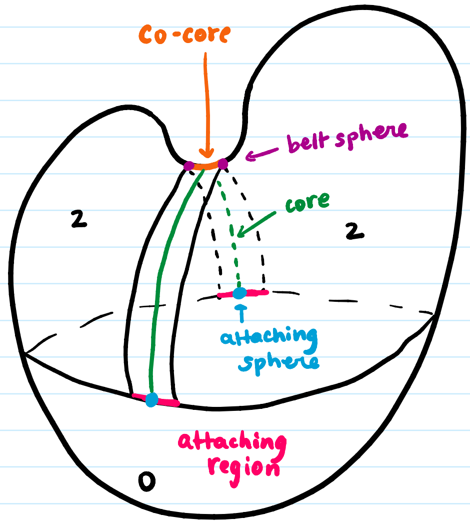
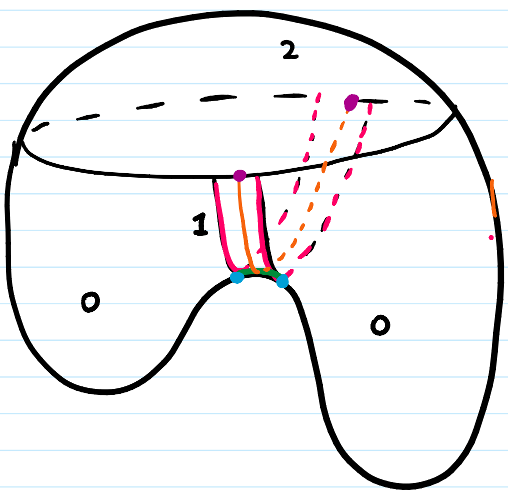
Week 7, 10/3 and 10/5

Tuesday, October 3, 2023 12:29 PM

1-2 cancelling pair

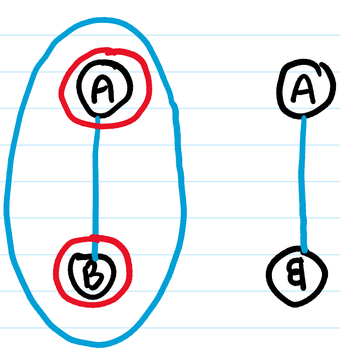


can flip upside down:



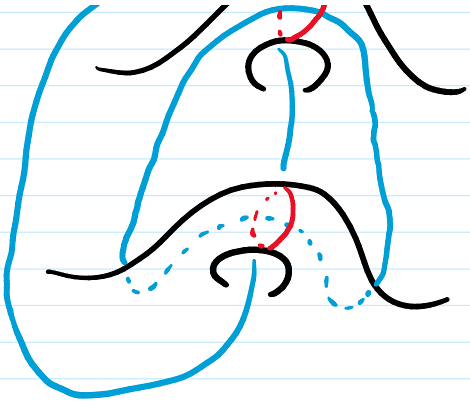
0-1 cancelling pair

Example:



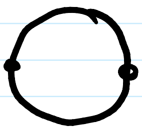
← Handlebody decomposition of a 3-mfd

Can think of this as being



this gives a genus 2 Heegaard splitting of some 3-mfd

Example: $T^2 = S^1 \times S^1$

$S^1 =$ 

$$= (0\text{-h}) \cup (1\text{-h})$$

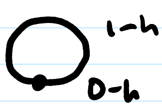
$$= \underbrace{D^1_-}_{0\text{-h}} \cup \underbrace{D^1_+}_{1\text{-h}}$$

$$S^1 \times S^1 = (D^1_- \cup D^1_+) \times (D^1_- \cup D^1_+)$$

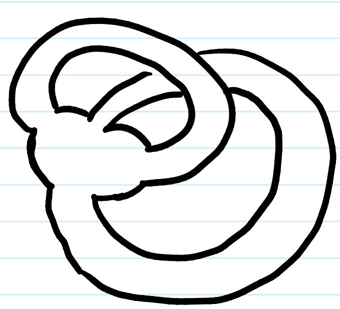
$$= D^1_- \times (D^1_- \cup D^1_+) \cup (D^1_+) \times (D^1_- \cup D^1_+)$$

$$= \underbrace{(D^1_- \times D^1_-)}_{0\text{-h}} \cup \underbrace{(D^1_- \times D^1_+)}_{1\text{-h}} \cup \underbrace{(D^1_+ \times D^1_-)}_{1\text{-h}} \cup \underbrace{(D^1_+ \times D^1_+)}_{2\text{-h}}$$

How are they attached to one another?

think of 

$\times D^1_-$ as  thickened $(0\text{-h}) \cup (1\text{-h})$

the other one is like  $\cup\ (2\text{-h})$

can redraw disk as being a half-plane

Example: $T^3 = S^1 \times S^1 \times S^1$

$T^2 = (0\text{-h}) \cup (1\text{-h}) \cup (1\text{-h}) \cup (2\text{-h})$

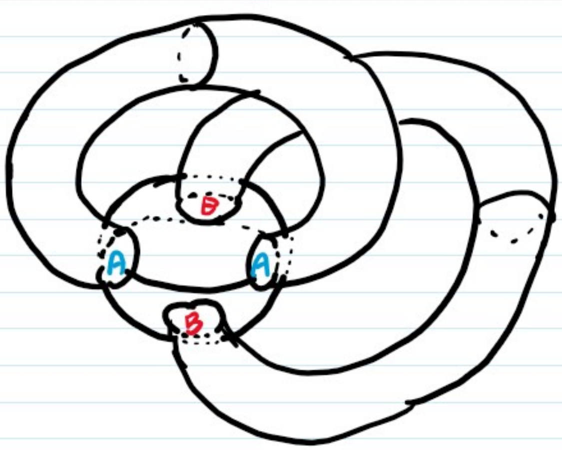
$S^1 = (0\text{-h}) \cup (1\text{-h})$

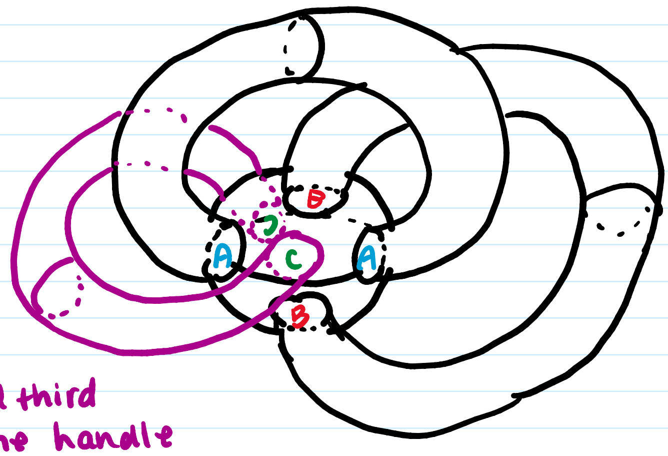
$T^2 \times S^1 = (0\text{-h}) \cup (1\text{-h}) \cup (1\text{-h}) \cup (1\text{-h}) \cup (2\text{-h}) \cup (2\text{-h}) \cup (2\text{-h}) \cup (3\text{-h})$

Crossing with 0-h is like crossing  with interval

thicken: 

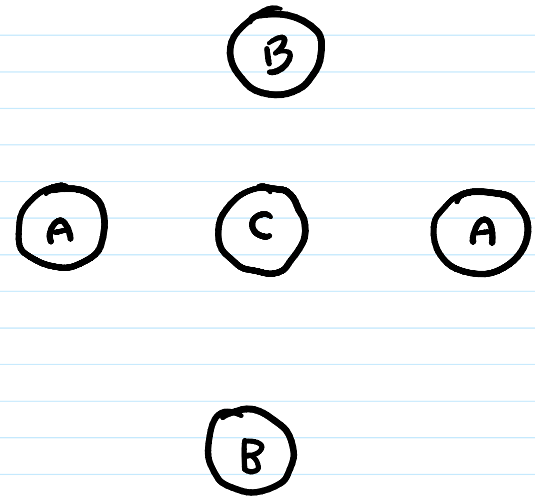
$\parallel$

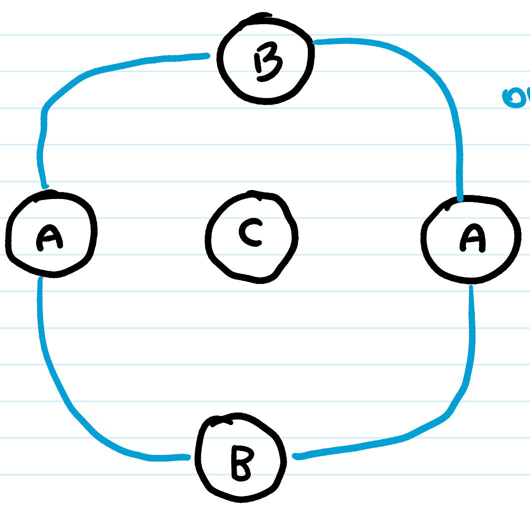




add third one handle

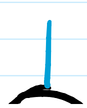
send back "C" to ∞

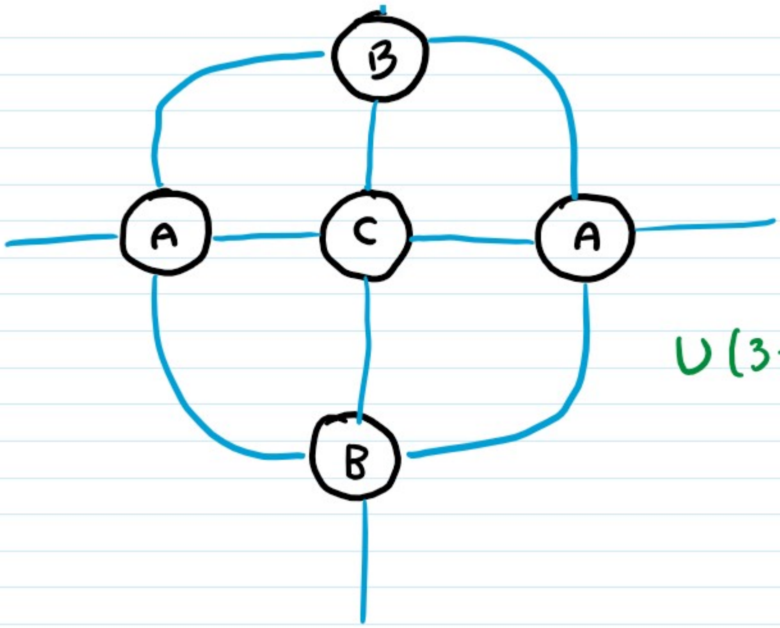




one of the two handles

add another 2-handle





$\cup$ (3-h)

Exercise:

Find a handle decomp. for $\Sigma_g \times S^1$

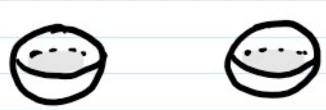
DIMENSION 4!

KIRBY DIAGRAMS

Unique 0-handle D^4, $\partial D^4 = S^3 = \mathbb{R}^3 \cup \{\infty\}$

from now on we just draw boundary & attaching region

The attaching region for 1-handles = $(\partial D^1) \times D^3 = D^3 \amalg D^3$



Attaching sphere for 2-handles = $(\partial D^2) \times \{0\} = S^1 \times \{0\}$

Attaching sphere for 2-handles = $(\partial D^2) \times \{0\} = S^1 \times \{0\}$

and need to specify a framing

What about the 3- and 4-handles?

In a closed 4-mfd, we can assume there is one 4-handle

Duality gives us that the union of the 3- and 4-handles will be 0-h $\cup$ some (1-h)'s which is $\natural_m S^1 \times D^3$

$m = \#$ of 3-handles

$$\partial\left(\natural_m S^1 \times D^3\right) = \#_m S^1 \times S^2$$

$\Rightarrow \partial$ (union of 0-, 1-, and 2-handles) $= \#_m S^1 \times S^2$

Theorem (Laudebach - Poénaru)

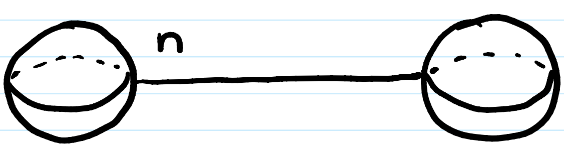
Any self-diffeo of $\#_m S^1 \times S^2$ extends over $\natural_m S^1 \times D^3$

Consequences:

1.) Need boundary of $(0\text{-h}) \cup (1\text{-h})\text{'s} \cup (2\text{-h})\text{'s}$ to be $\#_m S^1 \times S^2$

2.) We don't need to specify the 3- or 4- handles (multiple ways of attaching gives us the same 4-mfd)

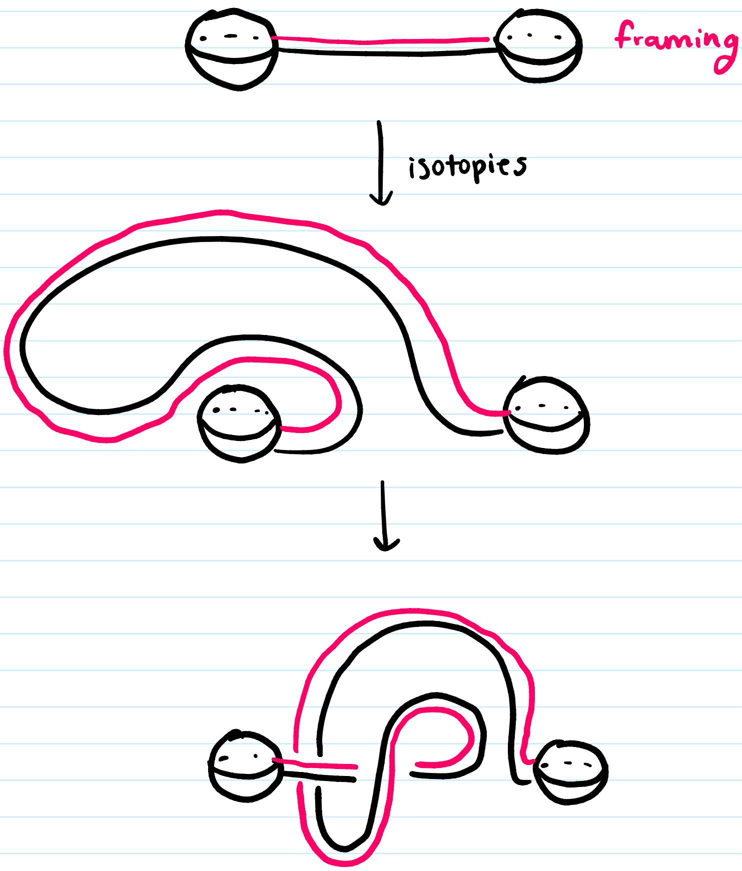
Example:



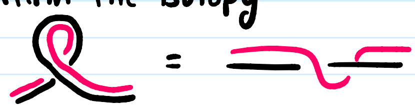
$\cup$ (4-h)

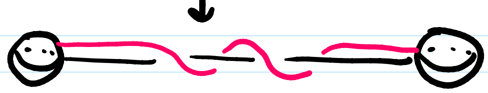
S^4

Note:

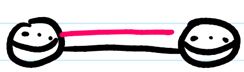


Exercise: confirm the isotopy





troublesome: isotopy gives up that

 equivalent to above

Framings of 2-handles that go over a 1-handle have an ambiguity.

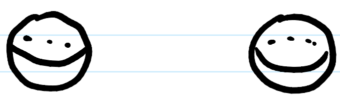
<u>Remark:</u> Later, we will describe a way to resolve this ambiguity

<u>**Example:**</u>

$S^3 \times S^1$ $\quad S^3 = 0\text{-h} \cup 3\text{-h}$

$S^1 = 0\text{-h} \cup 1\text{-h}$

$S^3 \times S^1 = 0\text{-h} \cup 1\text{-h} \cup 3\text{-h} \cup 4\text{-h}$



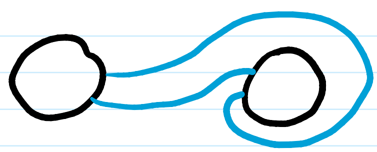
$\cup (3\text{-h}) \cup (4\text{-h})$

Example:

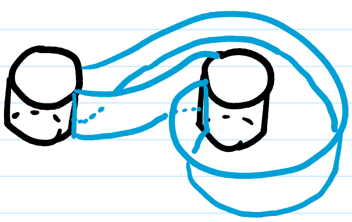
$\mathbb{RP}^3 \times S^1$

$\mathbb{RP}^3 = 0\text{-h} \cup 1\text{-h} \cup 2\text{-h} \cup 3\text{-h}$

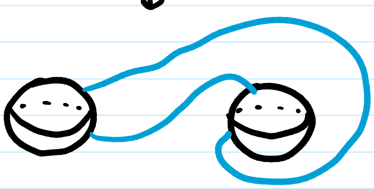
$S^1 = 0\text{-h} \cup 1\text{-h}$



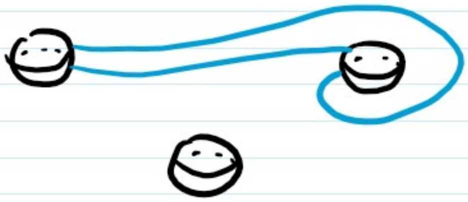
$\mathbb{RP}^3 \times S^1 = (0\text{-h}) \cup 2(1\text{-h})\text{'s} \cup 2(2\text{-h})\text{'s} \cup 2(3\text{-h})\text{'s} \cup (4\text{-h})$



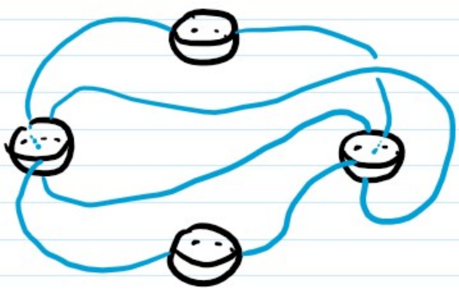
$\downarrow$



$\downarrow$ add another one handle



add 2-handle



$\cup\, 2\,(3\text{-h})$'s $\cup\, (4\text{-h})$

DOUBLES:

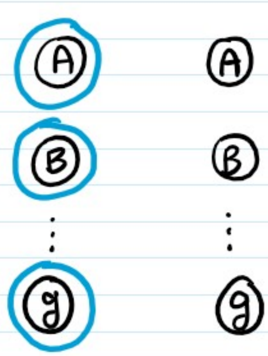
M^n mfd with boundary. Can form the double of M

$$DM = \partial(I \times M) = M \cup_{id_\partial} -M$$

Ex: $DD^n = S^n$

Ex: $D(\Sigma_g - \mathring{B}^2) = \Sigma_{2g}$

Ex: $DH_g = \#_g\, S^2 \times S^1$



Ex: X = 4-mfd with no 3- or 4- handles

$DX = X \cup$ handles

each 2-h of $X \rightsquigarrow$ new 2-h

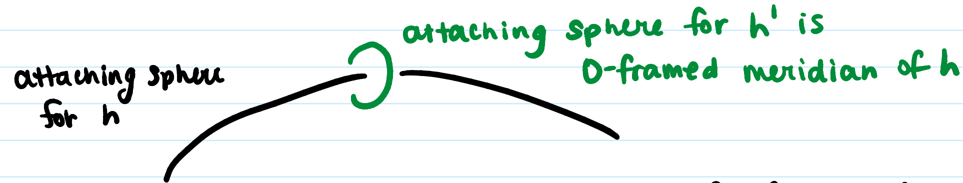
each 1-h of $X \rightsquigarrow$ new 3-h

0-h of $X \rightsquigarrow$ 4-h

Sufficient to understand up to (2-h)s

- each new 2-handle h' is a copy of an old 2-handle h with core and cocore interchanged

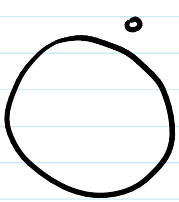
  h' attached to ∂X along belt sphere of h



see Ex. 4.6.3 of Gompf and Stipsicz for the framing explanation.

$\Rightarrow$ double DX of X is obtained by adding a 0-framed meridian to each 2-handle of X, then attaching 3 and 4-handles.

Examples:



$S^2 \times D^2$

$(S^2 \times D^2) \cup (S^2 \times D^2)$

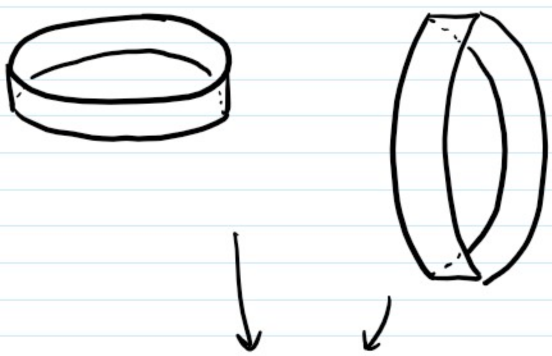
$(S^2 \times D^2) \cup (S^2 \times D^2)$

exercise: How are they identified?

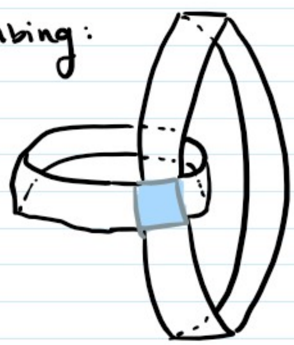
$D_1 \times D^2 \sim_f D_2 \times D^2$

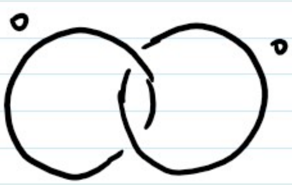
swaps the factors (fiber $\longleftrightarrow$ base)

lower dimensional analog: plumbing



plumbing:





$\cup$ (4-h)

boundary has matrix

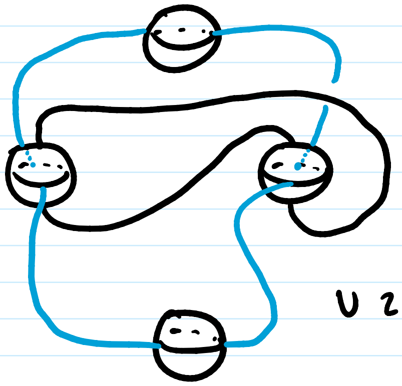
$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

so ∂ must be S^3

So this is $S^2 \times S^2$.

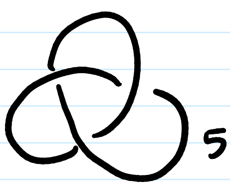
Example:

$\mathbb{R}P^3 \times S^1$



$\cup\ 2\ (3\text{-h})$'s $\cup\ (4\text{-h})$

Ex: 4-mfd X with boundary X

X 

$DX =$ 

Exercise: Given a Kirby diagram for a 4-mfd, find a presentation for $\pi_1(X)$

Question: Can we read off the intersection form of a 4-mfd from a handle decomposition / Kirby diagram?

Remark: Smooth 4-mfds with boundary have an intersection form

$$Q_X : H_2(X) \otimes H_2(X) \longrightarrow \mathbb{Z}$$
$$(a,b) \longmapsto a \cdot b$$

signed count of intersection points between sm.

points between sm.
surfaces represent-
ing a and b

- symmetric, integral, bilinear form.
- can't do Poincaré duality in the same way when X is not closed
  ↳ Q_X need not be unimodular

Example: $S^2 \times D^2 \quad Q_X = 0$

Let R be a ring
An R-homology sphere M^n is a manifold with

$$H_*(M^n; R) = H_*(S^n; R)$$

We will mostly care about $R = \mathbb{Z}$ and $n = 3$

integer homology sphere

Ex: S^3 is the simplest example of an $\mathbb{Z}HS^3$

Ex: Poincaré Homology sphere

$S^3_{-1}($  $)$ $\qquad \Sigma(2,3,5)$

(and $\Sigma(\underbrace{p,q,r}_{\text{rel. prime}})$)

Ex: #'s sums of these

Ex: $S^3_{1/n}(K) \quad \forall\, n \in \mathbb{Z}$ and any knot K

Ex: $S^3_{p/q}(K)$ $\begin{matrix} p \neq \pm 1 \\ p \neq 0 \end{matrix}$ is a $\mathbb{Q}H_*S^3$ is a

rational homology sphere but not an $\mathbb{Z}HS^3$.

Exercise: X a 4-mfd with boundary.

Q_X is unimodular $\iff$ ∂X is a disjoint union of $\mathbb{Z}HS^3$'s

Theorem:

Let X be a 4-mfd with no 1 or 3 handles, described by a framed link $\mathcal{L}$. Then

$$Q_X = \text{linking matrix}$$

proof idea:

Consider $X_n(K)$ = n-trace of K

↳ take n-framed 2-handle attached to K

Generator for $H_2(X_n(K))$: $\hat{F}$

take Seifert surface F for K

$\hat{F} = (F \text{ pushed into } B^4) \cup (\text{core of 2-h})$

What is $\hat{F} \cdot \hat{F}$?

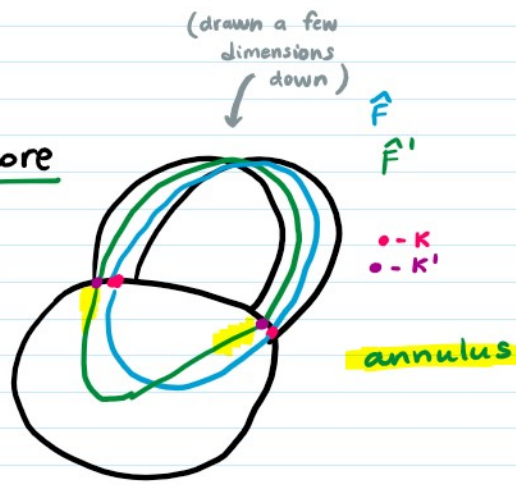
$\hat{F}'$ = parallel copy of core

$\cup$

annulus diving into B

$\cup$

F pushed (deeper) into B^4



Exercise: $\hat{F} \cdot \hat{F}' = n$

The parallel copy of core intersects S^3 in K'

having framing n means $lk(K, K') = n$

Idea: $lk(K, K') = n = F \cdot K'$ in S^3

Now for a framed link:

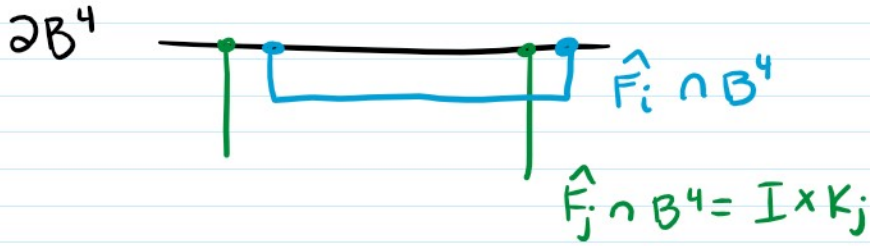
$n_i \cdot$ framed $K_i \rightsquigarrow \hat{F}_i$

$i=j$: $\hat{F}_i \bullet \hat{F}_i = n_i$ as before

$i \neq j$: Assume $\hat{F}_j$ deeper in B^4 than $\hat{F}_i$

i.e. $\hat{F}_i \cap B^4$ lives in a collar $I \times S^3$ of ∂B^3

$\hat{F}_j$ vertical in this collar



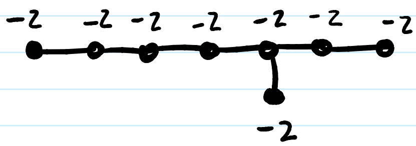
$$\hat{F}_i \cdot \hat{F}_j = F_i \cdot K_j = lk(K_i, K_j)$$

///

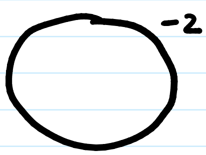
Example:

$\cup\, (4\text{-}h)$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

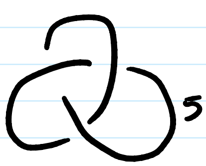
Ex: $E_8 =$ 

Simply connected 4-mfd X whose ∂ is Poincaré Homology Sphere and has intersection form $Q_X = E_8$
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ ‖
plumbing of D^2-bundles over S^2

Ex: 

D^2 bundle over S^2

Ex:

X 

$Q_X = [5]$

$DX =$ 

$Q_{DX} = \begin{bmatrix} 5 & 1 \\ 1 & 0 \end{bmatrix}$

Q: When do two Kirby diagrams represent the same 4-mfd?

A: <u>KIRBY CALCULUS</u>

Theorem: (Cerf)

> Given any two handle decompositions for a compact mfd M^n, it's possible to get from one to another by a sequence of handleslides, creating/annihilating cancelling pairs of handles, and isotopies (within levels)

proof:

1-parameter families of Morse functions.

Compare to Heegaard moves (isotopies, handleslides, stabilization/destabilization)

Cancelling Pairs:

$(k-1)$-handle h_{k-1}

k-handle h_k

attaching sphere of h_k intersects belt sphere of h_{k-1} transversely at a single point

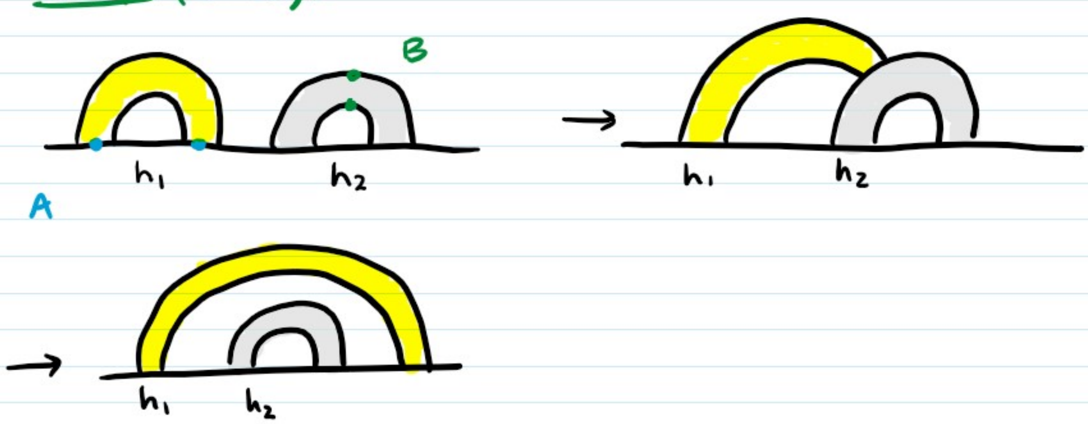
Handleslide:

two k-handles h_1, h_2 with indices k_1, k_2 $(0<k<n)$ attached to ∂X

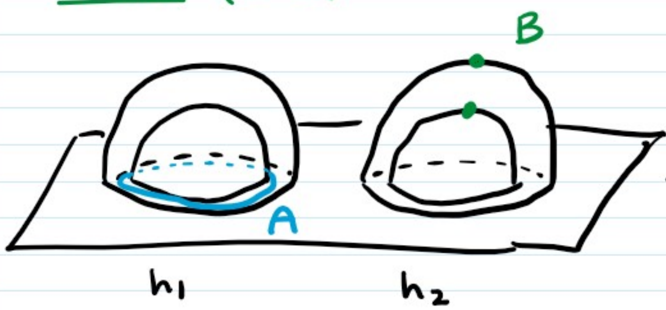
A handleslide of h_1 over h_2

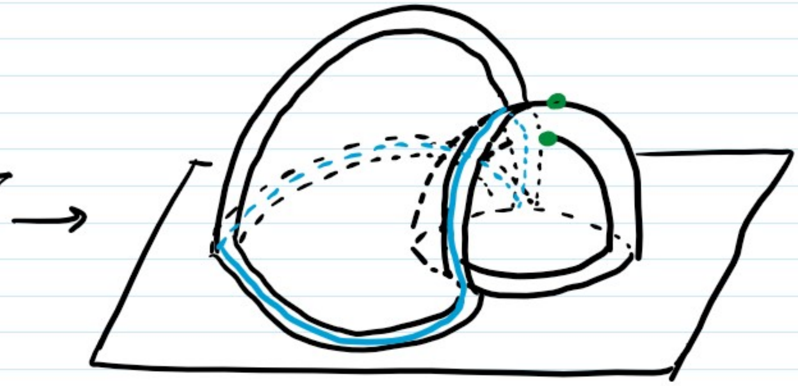
- isotope attaching sphere A of h_1 in $\partial(X \cup h_2)$ pushing it across belt sphere B of h_2

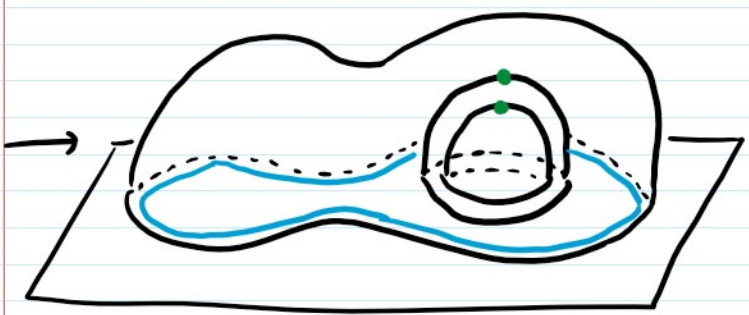
Example (dim. 2):



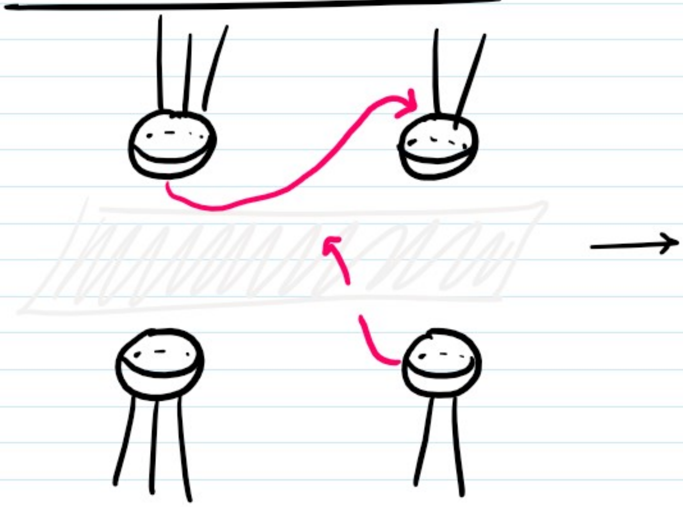
Example: (dim 3)

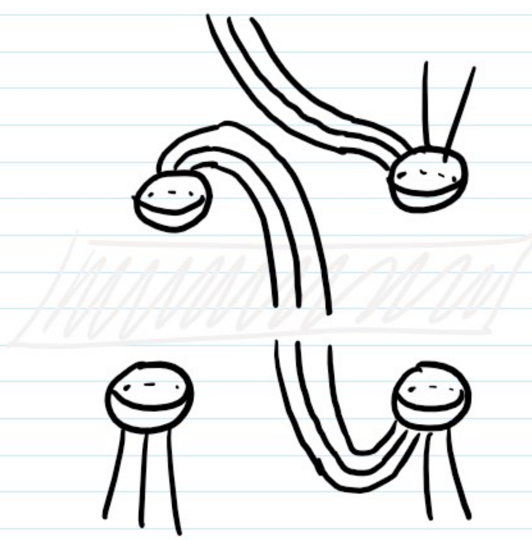






4-dim 1-handleslide:





Can use double strand notation to keep track of the framing!