L1 6^{th}, Jan, 2025.

Goal: study knot concordance & homology cobordism groups.

- understand the structure of these structures.
- applications to other areas of topology

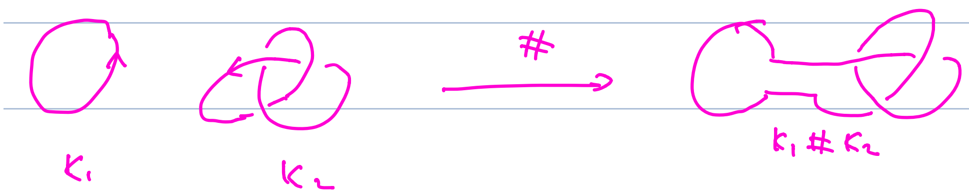
Now we'll consider knots in S^3.

Two knots K_1, K_2 are equiv$^{=}$ if $\exists$ ori$^{=}$ preserving homeo$^{=}$ $\varphi: S^3 \to S^3$ s.t. $\varphi(K_1) = K_2$.

Fact: knots $K_1, K_2 \subseteq S^3$ are equi$^{=}$ iff they are ambient isotopic.

Remark: in arbitrary 3-manifold Y, equiv$^{=}$ is weaker than ambient isotopy if $MCG(Y)$ is not trivial.

Connect sum on knots:



$\boxed{Ex}$ a) # is well-defined.

b) # is commutative.

Natural questions about $(\{\text{knots in } S^3\}, \#)$:

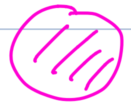
identity? Yes, the unknot.

inverse? No, except K = unknot.

This turns to a commut$^{=}$ monoid

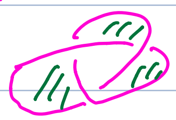
Def: A Seifert surface for a knot K is a compact, oriented, connected surface F s.t. $\partial F = K$.

Ex $g=0$ $g=1$



F = disk when K = unknot

nonexample:



is not orientable

Such surface is characterized by genus

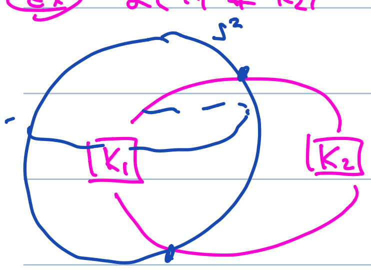
Def $g(K) = \min\{\text{genus of } F \mid F \text{ is Seifert surface of } K\}$.

Ex ① $g(\text{unknot}) = 0$ in fact, $g(K)=0 \iff K$ is unknot.

② $g(\text{trefoil}) = 1$.

More generally,

Ex $g(K_1 \# K_2) = g(K_1) + g(K_2)$.



Consider pt of intersection, this gives $\geq$.

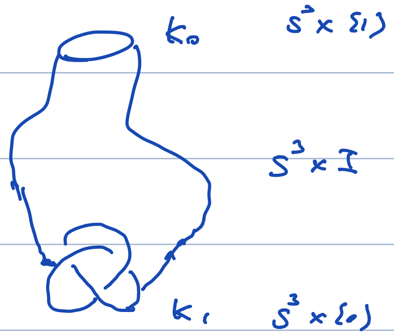
Consequence:

$g(K_1 \# K_2) = 0 \iff K_1 = K_2 = \text{unknot}$ since genus ≥ 0.

In other words, we don't have inverses in our monoid.

Def: Knots K_0, K_1 in S^3 are smoothly concordant, $K_0 \sim K_1$, if they cobound a smoothly embedded annulus in $S^3 \times I$

Ex "$\sim$" is an equiv$^{\underline{c}}$ relation.

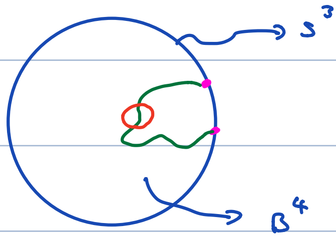


← picture

(topologically, locally flat) every pt has product nbhd.

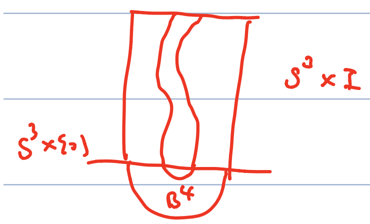
<u>Def</u>: A knot is smooth / topological slice if $K \sim$ unknot

<u>Note</u>: A knot is smoothly slice $\Leftrightarrow$ K bounds a smoothly embedded disk in B^4.



Let the pink pts be two knots.

$\Leftarrow$



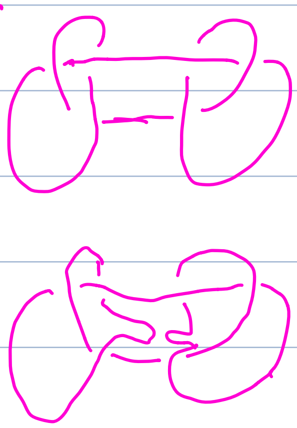
$\Rightarrow$

//

If we remove all adjectives and only require disk to be embedded, then every knot would be slice.

$\mathrm{Cone}(S^3, K) = (B^4, D^2)$.

<u>Ex</u>

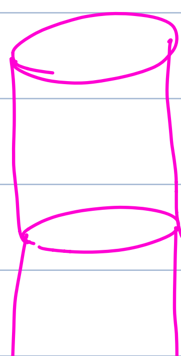


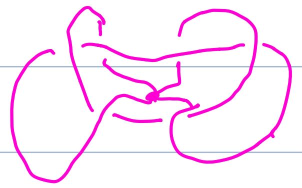
$S^3 \times \{1\}$

$\downarrow$

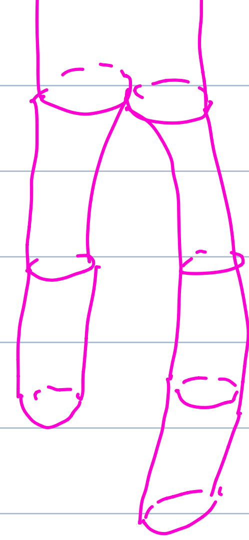
$S^3 \times \{9/10\}$

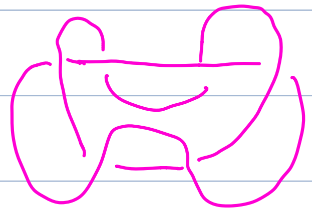
$\downarrow$



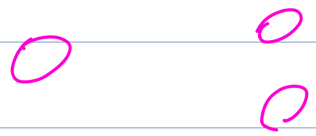


$S^3 \times \{8/10\}$



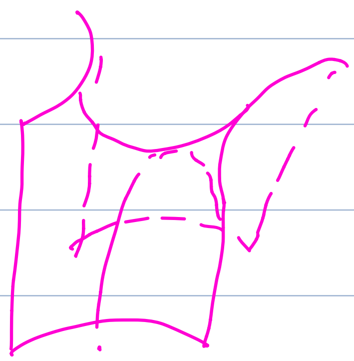


$S^3 \times \{1/2\}$



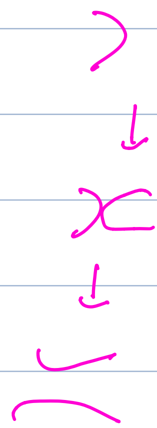
$S^3 \times \{0\}$

An analogy:



Saddle.

The projⁿ looks like



etc

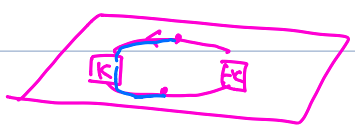
Mirror of K: switch under / over crossing. mK

reverse of K: switch orientations. K^r

$mK^r = -K$.

Prop: $K \# -K$ is smoothly slice.

Pf.



S^3

The pts are connect sum

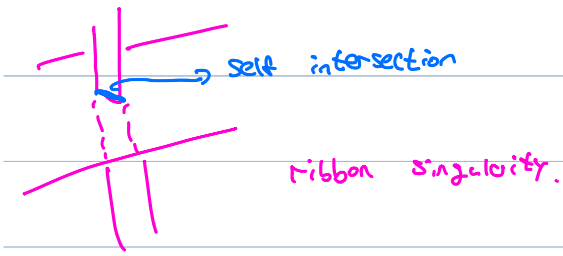
Sweeping blue arc gives a disk. //

Def: A knot K in S^3 is ribbon if it bounds a smoothly embedded disk D^2 in B^4, with no local max wrt radial morse function on $f: B^4 \to \mathbb{R}$.

Ex

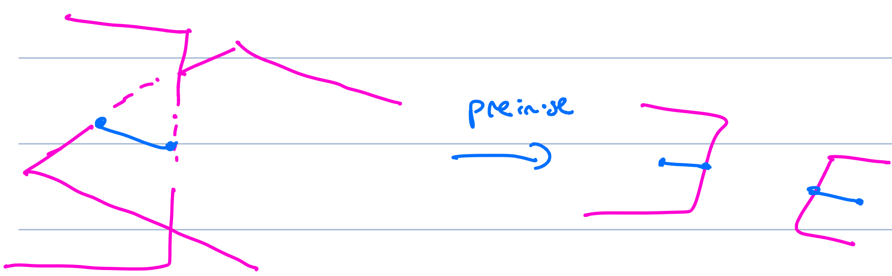
so this is not ribbon.

Ex K is ribbon $\iff$ K bounds an immersed disk with only ribbon singularities.



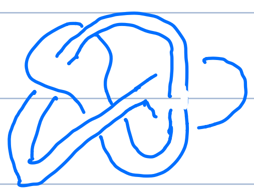
Preimage of this arc has 2 arcs. one lies entirely in the interior of the surface. the other has 2 endpts in the boundary

By contrast:



not ribbon

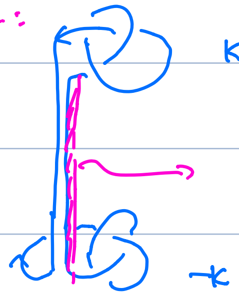
Ex



This is ribbon

In fact, $K \# -K$ is ribbon

Picture:



K = trefoil.

sweep this around gives the disk.

$-K$

observe: ribbon $\Rightarrow$ slice.

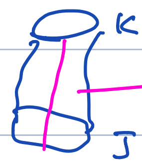
Q: is every slice ribbon? This leads to

Slice - ribbon Conjecture.

L2. 8^{th}, Jan, 2025

Ex $K \sim J \iff K \# -J$ slice

($\Rightarrow$)

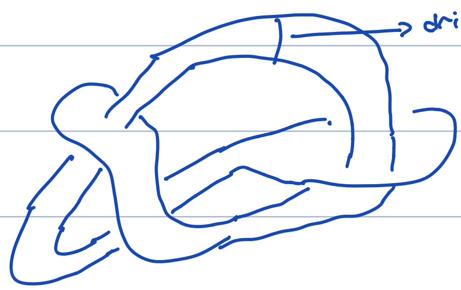


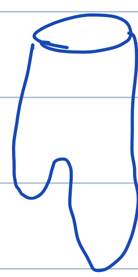
drill nbhd of this arc.

Fact: slice ribbon conjecture is true for 3-strand pretzel knot $P(p,q,r)$ with p,q,r odd.

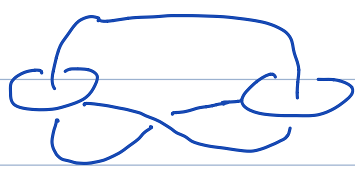
Ex: show that $P(p,-p,r)$ is odd.

Ex





'dual'



Knot concordance group.

$$\mathcal{C} := \left(\{\text{Knots in } S^3\}/\sim_{\text{concordance}} \, , \; \# \right)$$

This is an abelian group with $\mathrm{id} = [\text{unknot}] = \{\text{slice knots}\}$

inverse of $[K]$ is $-[K] = [-K]$

Ex There is an isotopy: figure 8 knot.

To conclude this is of order 2, it suffices to show it's not slice

Seifert form & the algebraic concordance group:

Let F be a Seifert surface of a knot K.

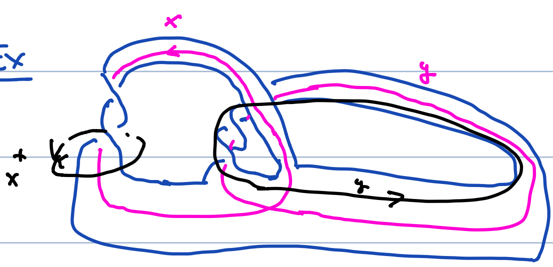
Ex Show the following are isotopic.

The Seifert form for K wrt F is the bilinear form

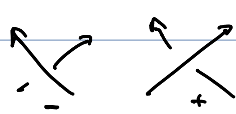
$$V_F : H_1(F;\mathbb{Z}) \times H_1(F;\mathbb{Z}) \longrightarrow \mathbb{Z}$$

$(x,y) \longmapsto lk(x, y^+)$, y^+ is the positive push-off of y.

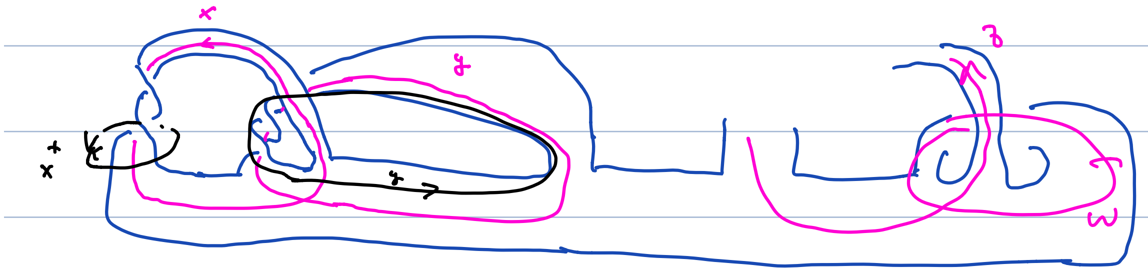
Ex



$$\begin{array}{c|cc} & x^+ & y^+ \\ \hline x & & \\ y & & \end{array} \quad \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix} = V.$$



If we consider a slightly more complicated Seifert surface of the same knot



$$\begin{array}{c|cccc} & x^+ & y^+ & z^+ & w^+ \\ \hline x & -1 & 1 & * & 0 \\ y & 0 & 1 & * & 0 \\ z & * & * & * & 1 \\ w & 0 & 0 & 0 & 0 \end{array}$$

This is called the stabilization.

Add 2 bands : one of which is untwisted, unknotted

the other can be twisted, knotted, link the original bands.

Effect on Seifert matrix:

$$V \longrightarrow \left[\begin{array}{c|cc} V & * & 0 \\ \hline * & * & 1 \\ 0 & & 0 \end{array}\right] \qquad (1)$$

<u>Handle slide</u> : Slide end of one band over an adjacent band.

Effect on Seifert matrix : $V \longrightarrow M V M^T$, M is invertible integer matrix. (2)

Ex Explicitly describe M.

<u>Def</u> Two integer matrices are S-equivalent if they are related by a sequence of operations ① and ②, and their inverse operations.

<u>Thm</u> : Any two Seifert surfaces for a knot K have a common stabilization.

<u>Coro</u> : Any two Seifert matrices of a knot are S-equivalent.

Consequence: Any invariant of a Seifert matrix V that only depends on S-equiv$^{\cong}$ class is a knot invariant.

Ex Alexander polynomial $\Delta_K(t) := \det(V - tV^T)$ up to a multiple of $\pm t^n$.

Ex a) This depends on the S-equiv$^{\cong}$ classes of V.

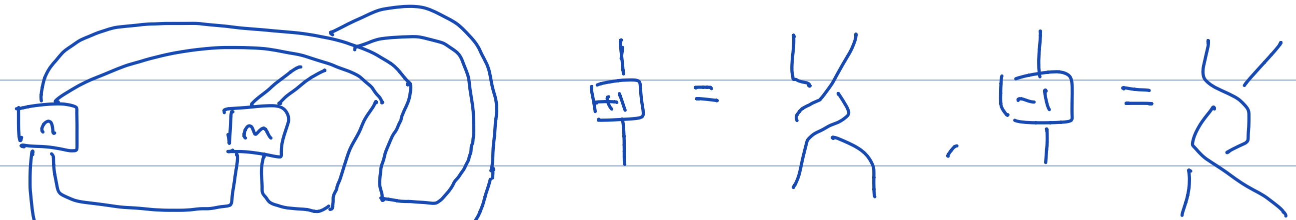
b) let $K =$ trefoil, $\Delta_K(t) = t^2 - t + 1$.

Ex Knot signature. $\sigma(K) = \mathrm{sgn}(V + V^T) \leftarrow$ (# of +ve eigenvalues − # of −ve eigenvalues)

Ex a) $\mathrm{sgn}(V + V^T)$ only depends on S-equiv$^{\cong}$ class of V.

b) σ(Right-handed trefoil) $= -2 = -\sigma$(LH trefoil).

Ex



etc.

Prove that if V_1, V_2 are Seifert matrices for K_1, K_2, then $V_1 \oplus V_2$ is a Seifert matrix for $K_1 \# K_2$. Conclude that $\Delta_{K_1 \# K_2}(t) = \Delta_{K_1}(t)\,\Delta_{K_2}(t)$, and that $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$.

Ex: If V is a Seifert matrix for K, then $-V$ is that of $-K$.

Q: What part of the Seifert matrix is concordance invariant?

Q: Do slice knots have 'nice' Seifert form.?

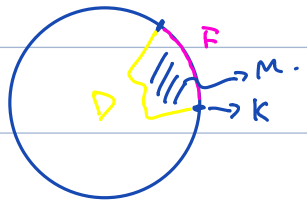
Prop: If K is topologically slice, F is any Seifert surface of K. Then $\exists$ a basis for $H_1(F; \mathbb{Z})$ s.t.

$$V = \begin{bmatrix} A & B \\ C & 0 \end{bmatrix}$$ A, B, C are of the same size, ($g \times g$ matrix)

(V is metabolic, i.e. vanishing in half dim subspace).

Idea of pf: If K is topological slice, with slice disk D in B^4, and Seifert surface F.

Picture.



$F \cup D$ bounds a 3-manifold M in B^4.

We need:

Lemma: If M is a compact, connected, oriented 3-manifold s.t. ∂M is a connected surface of genus g, then $\exists$ basis of $H_1(\partial M, \mathbb{Q})$ s.t. under the map $i_*: H_1(\partial M, \mathbb{Q}) \longrightarrow H_1(M, \mathbb{Q})$ with a basis $x_1, \cdots, x_g, y_1, \cdots, y_g \in H_1(\partial M, \mathbb{Q})$ s.t. $x_i \in \ker(i_*)$ and $y_i \notin \ker(i_*)$ $\forall i$.

To prove the lemma, use les of $(M, \partial M)$. w/ $\mathbb{Q}$-coefficients.

There are iso$^{=}$ $H_3(M, \partial M) \longrightarrow H_2(\partial M)$, $H_0(\partial M) \longrightarrow H_0(M)$, so

$$0 \to H_2(M) \to H_2(M, \partial M) \to H_1(\partial M) \xrightarrow[i_*]{} H_1(M) \to H_1(M, \partial M) \to 0.$$

By PD, $H_1(M, \partial M) \cong H^2(M)$, $\dim H^2(M) = \dim H_2(M)$.

Ex $\dim H_2(M, \partial M) = \dim H_1(M)$.

Then By exactness & rank-nullity to conclude $\dim(\operatorname{im}(i_*)) = \dim(H_1(M)) - \dim(H_2(M))$.

and similarly $\dim(\ker(i_*)) = \dim(H_1(M)) - \dim(H_2(M))$. //

Back to prop. $H_1(F) \cong H_1(F \cup D^2) = H_1(\partial M)$.

Ex Conclude the pf by the following alternative def of linking number.

Def $J, K \subset S^3 = \partial B^4$. then $lk(J, K)$ is the signed intersection # of A, B, where A, B are 2-chains in B^4. s.t. $\partial A = J$, $\partial B = K$. //