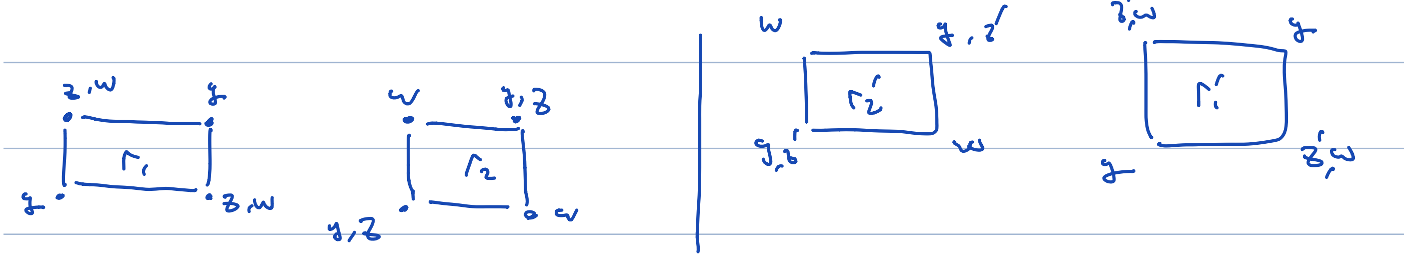
L18 Mar 10 2025

Prop: $d^2_{\mathbb{X},\mathbb{O}} = 0$.

Pf: Fix $y \in S(G)$, then $d^2_{\mathbb{X},\mathbb{O}}(y) = \partial_{\mathbb{O},\mathbb{X}}\left(\sum_{z \in S(G)} \# \operatorname{Rect}^\circ_{\mathbb{X},\mathbb{O}}(y,z)\cdot z\right)$

$$= \sum_{z \in S(G)} \# \operatorname{Rect}^\circ_{\mathbb{X},\mathbb{O}}(y,z)\left(\sum_{w \in S(G)} \# \operatorname{Rect}^\circ_{\mathbb{X},\mathbb{O}}(z,w)\right) w$$

Case 1: Corners of r_1, r_2 are all distinct

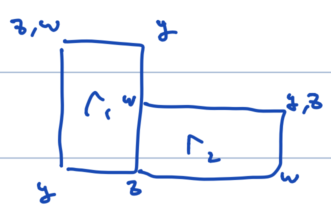


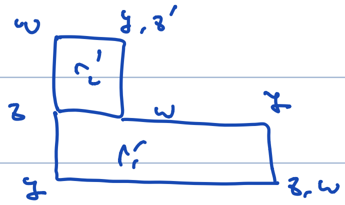
Then $\exists!\ z' \in S(G)$ and rectangle $r_1' \in \operatorname{Rect}^\circ_{\mathbb{X},\mathbb{O}}(y,z')$, $r_2' \in \operatorname{Rect}^\circ_{\mathbb{X},\mathbb{O}}(z',w)$

s.t. r_1, r_2 and r_1', r_2' have same support.

r_1, r_2 contribute a w term to ∂^2, so are r_1', r_2'. Thus the result follows.

Case 2: r_1, r_2 share a corner.





$r_1 \in \operatorname{Rect}^\circ_{\mathbb{X},\mathbb{O}}(y,z)$

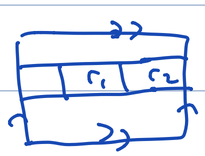
$r_2 \in \operatorname{Rect}^\circ_{\mathbb{X},\mathbb{O}}(z,w)$

$r_1' \in \operatorname{Rect}^\circ_{\mathbb{X},\mathbb{O}}(y,z')$

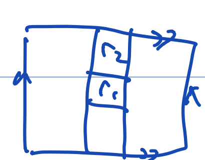
$r_2' \in \operatorname{Rect}^\circ_{\mathbb{X},\mathbb{O}}(z',w)$

$\exists!\ z' \in S(G)$, r_1', r_2' s.t. $r_1 \cup r_2 = r_1' \cup r_2'$ so the result follows.

Case 3: r_1, r_2 share an edge.



or



But this cannot happen, since r_1, r_2 are totally empty but each row contains exactly one X and one O. //

Idea behind invariance:

Show invariance of $\widehat{GH}$ under commutation & (de)stabilization

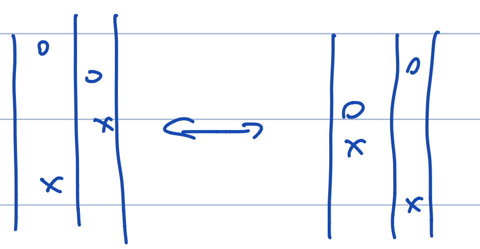
Commutation: suppose G, G' differ by commutation.

want to construct chain maps $P : \widetilde{GC}(G) \longrightarrow \widetilde{GC}(G')$ and $P' : \widetilde{GC}(G') \to \widetilde{GC}(G)$ and chain homotopy:

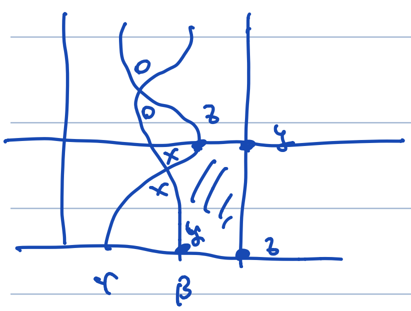
$H : \widetilde{GC}(G) \to \widetilde{GC}(G)$, $H' : \widetilde{GC}(G') \to \widetilde{GC}(G')$ s.t.

① $P' \circ P + id_{\widetilde{GC}(G)} = \partial^0_{*,0} H + H \partial^0_{*,0}$

② $P \circ P' + id_{\widetilde{GC}(G')} = \partial^0_{*,0} H' + H' \partial^0_{*,0}$.



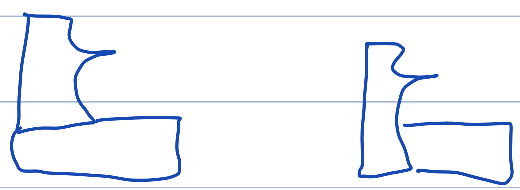
use β to get G, use γ to get G'



P counts pentagons.

P is a chain map

idea:



Analogous to pf that $\partial^2 = 0$

H counts hexagons, the pf of chain homotopy is similar //

$\widehat{HFK}(K) = \widehat{GH}(G)$, $\widetilde{GH}(G) \otimes W^{n-1} = \widehat{GH}(G)$.

$GC^-(G)$ = chain complex generated over $\mathbb{F}[u_1, u_2, \cdots, u_n]$ by $S(G)$
n×n grid

the corresponding differential is

$$\partial^-_{\mathbb{X}}(y) = \sum_{w \in S(G)} \sum_{r \in \mathrm{Rect}^\circ_{\mathbb{X}}(y,z)} \prod_{i=1}^{n} u_i^{n_{O_i}(r)} \, z$$

Here $\mathrm{Rect}^\circ_{\mathbb{X}}(y,z)$ = empty rectangles from y to z w/ no X inside

$n_{O_i}(r)$ = # of O_i appear in r (either 0 or 1)

O_i means we label the n o's by $O_1, \cdots, O_n$

u_i has | Alexander | grading | -2
| Maslov | | -1.

$\boxed{\text{Exercise: } \partial_{\mathbb{X}} \text{ preserves Alexander / Maslov gradings}}$

Prop multiplication by u_i and u_j are chain homotopic. i.e. $\exists$ map
$H: GC^-(G) \to GC^-(G)$ s.t. $u_i + u_j = \partial_{\mathbb{X}} H + H \partial_{\mathbb{X}}$.

'All the u_i collapse at the level of homology'.

Thus $H_*(GC^-(G))$ is a module over $\mathbb{F}[u]$

$HFK^-(K) := H_*(GC^-(G))$

Exercise $H_*\left(GC^-(G)/u_i=0\right) \otimes W^{n-1} = H_*(\widetilde{GC}(G))$.

so $H_*(GC^-(G)/u_i=0) = \widehat{HFK}(K)$.

Q: Can we get concordance invariance from knot Floer homology?

If we allow rectangles that contain X's, then the Alexander grading becomes a filtration

$$0 \subseteq \cdots \subseteq F_{s-1} \subseteq F_s \subseteq F_{s+1} \subseteq \cdots = GC^-(G)/_{U_i=0}$$

we have $A(\partial y) \leq A(y)$.

Moreover, the total homology wrt this differential is $\mathbb{F}$, so we can define

$$\tau(K) = \min\{s \mid F_s \hookrightarrow GC^-(G)/_{U_i=0} \text{ is surjective on } H_*\}$$

Heegaard Floer homology

Heegaard diagrams:

Def A handlebody of genus g is a closed, regular nbhd of $\bigvee_{i=1}^{g} S^1 \subseteq \mathbb{R}^3$

Def A Heegaard splitting of a 3-manifold Y is a decomposition of $Y = H_1 \cup_\varphi H_2$.

where H_1, H_2 are handlebodies, φ is orientation-reversing homeomorphism

The genus of the splitting is the genus of ∂H_1 or ∂H_2, $\partial H_1 = -\partial H_2 = \Sigma$

is the Heegaard surface.

Thm All 3-manifolds Y admits a Heegaard splitting.

Pf 1: Every 3-manifold admits a triangulation. H_1 = nbhd of 1-skeleton.

Then $H_2 = Y - H_1$. (Equivalently, H_2 = nbhd of dual 1-skeleton) //

L19 12th, Mar, 2025

Pf 2: let $f: Y \to \mathbb{R}$ be a self-index Morse function.

i.e. $f(c) = \text{index}(c)$ for each c critical pt.

Then $H_1 = f^{-1}((-\infty, 3/2])$, $H_2 = f^{-1}([3/2, \infty))$, $\Sigma = f^{-1}(3/2)$

A Heegaard splitting of a 3-manifold Y consists of a surface Σ of genus g

that bounds a handlebody on each side.

Week 10

Wednesday pg 2

Last time:

Heegaard splittings

$Y = H_1 \cup_\varphi H_2$ $\qquad H_1, H_2$ handlebodies

$\varphi : \partial H_1 \longrightarrow \partial H_2$

orient. reversing homeo.

Alternative Proof that every 3 mfd admits a Heegaard splitting.

Let $f : Y \longrightarrow \mathbb{R}$ be a self-indexing Morse function

i.e. $f(c) = \text{index}(c)$ for each critical point

Then $H_1 = f^{-1}\left((-\infty, \frac{3}{2}] \right)$ ⟵ index 0, 1 crit. pts

3-dim 1-handles

$H_2 = f^{-1}\left([\frac{3}{2}, \infty) \right)$ ⟵ dually a 3-handle and 2-handle

$\Sigma = f^{-1}(3/2)$

A Heegaard splitting of a 3-mfd Y consists of a surface Σ of genus g that bounds a handlebody on both sides

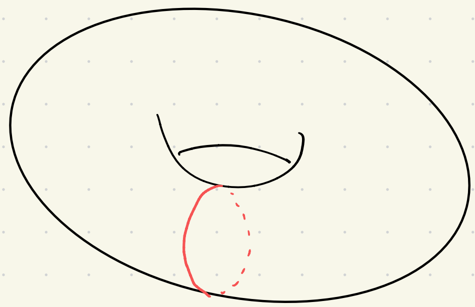
Let H be a handlebody of genus-g. A set of attaching curves for H is a set $\{\gamma_1, \ldots, \gamma_g\}$ of simple closed curves in ∂H s.t.

1. curves are pairwise disjoint

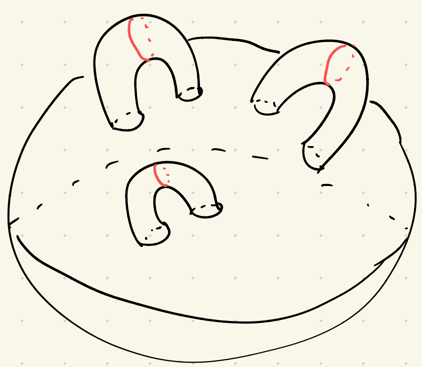
2. $\Sigma - \{\gamma_1, \ldots, \gamma_g\}$ connected

3. each γ_i bounds a disk in $\mathcal{H}$

Example: solid torus



Example:



boundaries of the co-cores of the 1-handles

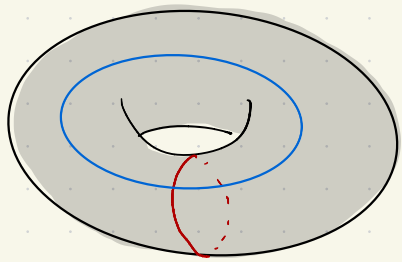
Attaching curves tell you how to fill in the surface
(glue in thickened disks along the attaching curves, can see S^2 in ∂ and ! way to glue in ball)

A Heegaard diagram compatible with a Heegaard splitting $Y = H_1 \cup_\varphi H_2$ is $\mathcal{H}(\Sigma, \alpha, \beta)$ where

1. Σ closed oriented surface of genus g
2. $\alpha = \{\alpha_1, \dots, \alpha_g\}$ attaching curves for H_1
3. $\beta = \{\beta_1, \dots, \beta_g\}$ attaching curves for H_2

Example:

S^3

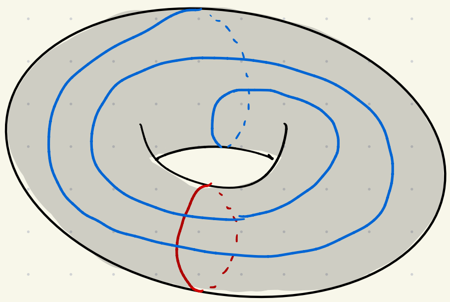


α always red

β always blue

Example:

$L(2,1)$



Lens space

$H_1(\text{3-mfd}) = \mathbb{Z}/2\mathbb{Z}$

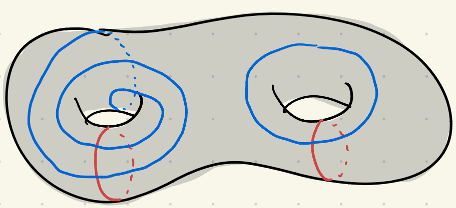
so this is $\mathbb{R}P^3$

Exercise: Describe how to compute $H_1(Y)$ from a Heegaard diagram for

(i.e. find a presentation matrix for $H_1(Y)$ from $\mathcal{H}$)

Example:

$\mathbb{R}P^3$



$= \mathbb{R}P^3 \# S^3$

Given a Heegaard diagram (Σ, α, β), we can build a 3-mfd as follows:

1. Thicken Σ to $\Sigma \times I$

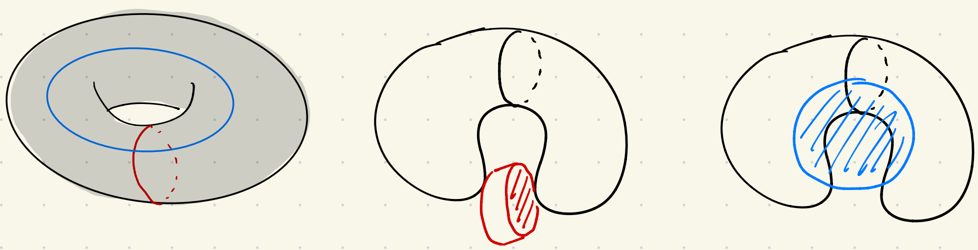
2. Along $\Sigma \times \{0\}$ attach thickened disks to $\alpha \times \{0\}$

3. Along $\Sigma \times \{1\}$ attach thickened disks to $\beta \times \{1\}$

   Exercise: The boundary of the resulting 3 mfd is $S^2 \amalg S^2$

4. Fill in each boundary component with B^3 (there is a unique way to do this since any orient. preserving homeo $S^2 \longrightarrow S^2$ is isotopic to id.)

Example:



α circles give co-cores of 1-handles.

β attaching circles for 2-handles

A Heegaard diagram gives a handlebody decomp in this way.

Goal:

Use Heegaard diagram to define a 3-mfd invariant

Q: When do two Heegaard diagrams describe the same 3-manifold?

Theorem:

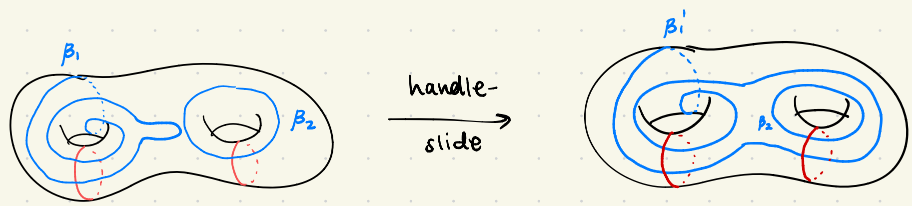
Two Heegaard diagrams describe the same 3-mfd

$\Longleftrightarrow$

they are related by a finite sequence of the following moves:

1. isotopy
2. handleslides

Example:

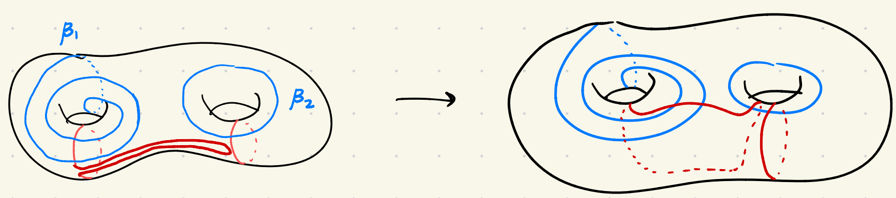


$\beta_1, \beta_2, \beta_1'$ cobound a pair of paints

doesn't change the handlebody that the β's are describing.

Can handleslide β_i over β_j and α_i over α_j

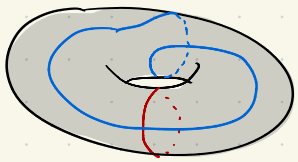
Doesn't change Heegaard splitting for this reason



sliding α_1 over α_2

3. stabilization/destabilization

connect sum with (T^2, α, β) where α and β are s.c.c. intersecting transversely in a single point

eg. 

For technical reasons, we will need a basepoint $w \in \Sigma$

Isotopies cannot cross w.

Handleslides cannot cross w

↳ w cannot be in the pair of pants cobounded by $\gamma_1, \gamma_2, \gamma_1'$, $\gamma = \alpha$ or β

Doubly Pointed Heegaard diagrams:

Defn: A doubly pointed Heegaard diagram for knot $K \subset Y$ is $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ where

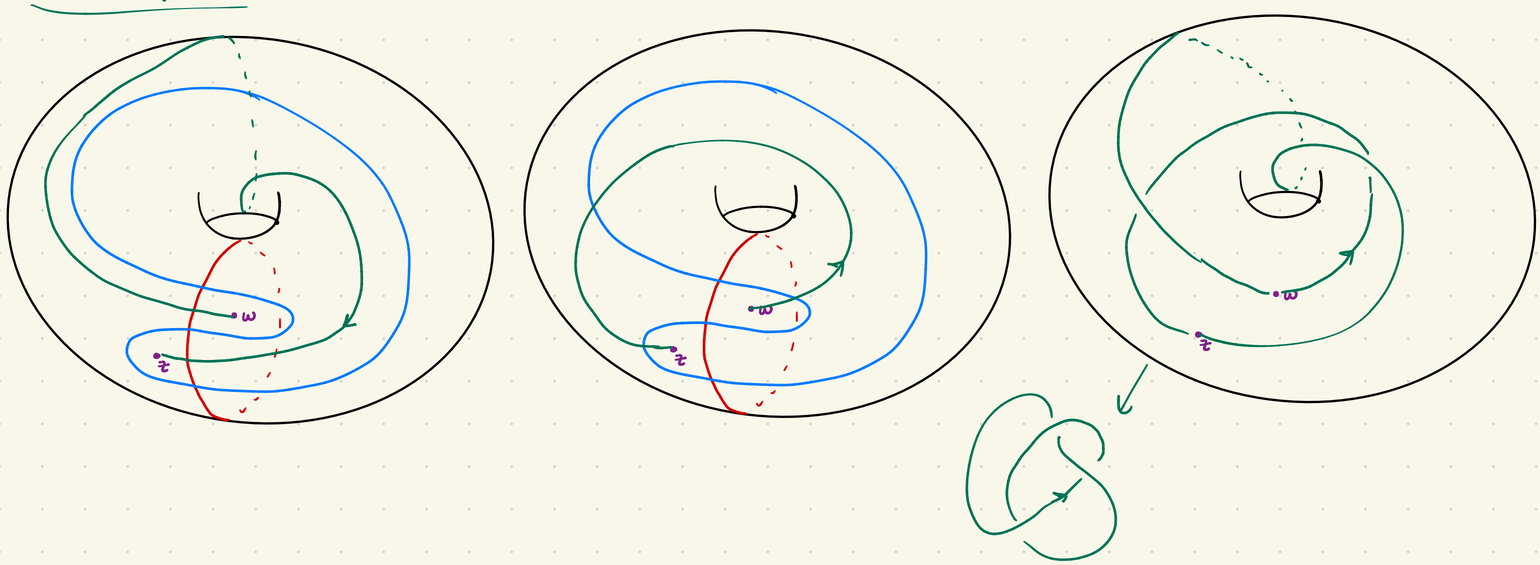
1. (Σ, α, β) Heegaard diagram for Y

2. K is the union of two arcs a and b where a is an arc in $\Sigma - \alpha$ connecting w to z pushed slightly into H_1

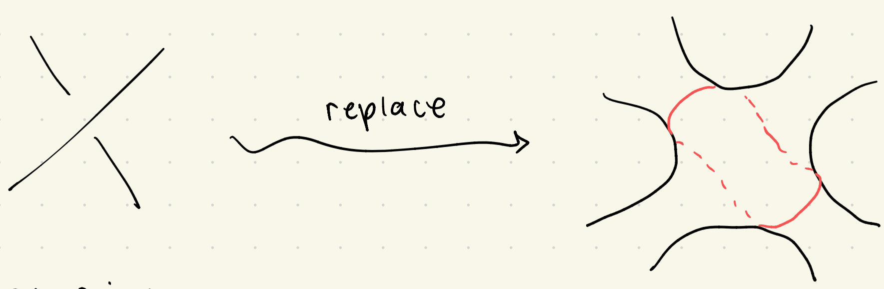
and β is an arc in $\Sigma - \beta$ connecting z to

w pushed slightly into H_2

Example:

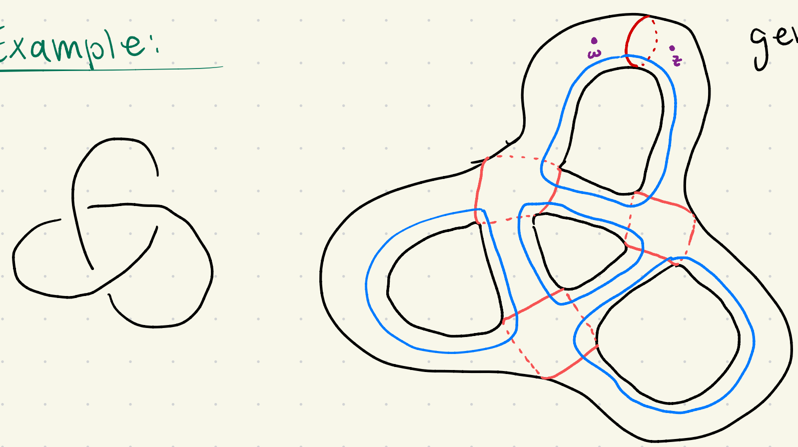


Given a knot diagram D for $K \subset S^3$, you can obtain a doubly pointed Heegaard diagram for K as follows:



crossing

Example:



genus 4 surface

Theorem

Two doubly pointed Heegaard diagrams represent the same knot iff they are related by a finite sequence of doubly pointed isotopies, handleslides, de/stabilizations

can't cross either point

More generally, let (Σ, α, β) consist of

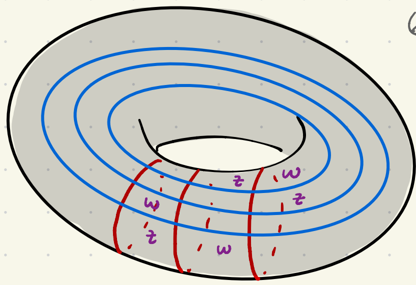
- a genus-g surface Σ
- $g+k$ disjoint s.c.c. $\alpha_1, \ldots, \alpha_{g+k}$ that span a half dimensional subspace of $H_1(\Sigma, \mathbb{Z})$
- $g+k$ disjoint scc's $\beta_1, \ldots, \beta_{g+k}$ span a half-dim subspace of $H_1(\Sigma, \mathbb{Z})$

Can build a 3-mfd roughly as before

1. Thicken Σ
2. Attach thickened disks to $\alpha_i \times \{0\}$ and $\beta_i \times \{1\}$
3. Attach $2(k+1)$ B^3's along resulting boundary cmpnts

Example:

S^3



$g=1$

$k=2$

this should look like a toroidal grid diagram

Now add basepoints $w_1, \dots, w_{k+1}$ and $z_1, \dots, z_{k+1}$

- each connected component of $\Sigma - \alpha_1 - \alpha_2 - \dots - \alpha_{g+k}$ contains exactly one w_i and z_i
- each connected component of $\Sigma - \beta_1 - \beta_2 - \dots - \beta_{g+k}$ contains exactly one w_i and z_i

This specifies a knot or link as before: connect w to z in $\Sigma - \alpha_1 - \dots - \alpha_{g+k}$ and push slightly into H_1 and connected z to w in $\Sigma - \beta_1 - \dots - \beta_{g+k}$ and push slightly into to H_2

Remark: Conventions for over/under crossings may differ from grid diagram conventions.

Heegaard Floer Homology :

Recall: grid states $S(\mathbb{G})$ were n-tuples of intersection points between vertical and horizontal circles using each circle exactly once

Heegaard Floer generators :

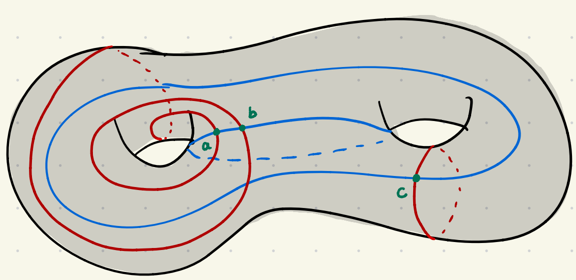
$(\Sigma, \alpha, \beta, w)$ $\quad\quad$ Σ genus g

$\alpha = \{\alpha_1, \dots, \alpha_g\}$

$\beta = \{\beta_1, \dots, \beta_g\}$

g-tuples of intersection points between α-circles and β-circles s.t. each α-circle (resp. β-circle) is used exactly once

Example:

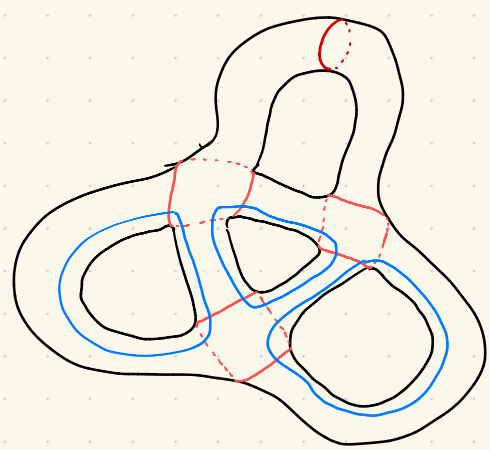


$x = \{a, c\}$

$y = \{b\}$

Exercise.

Find all the Heegaard Floer generators



Where do these generators come from?

$$\mathrm{Sym}^g(\Sigma) = \underbrace{\Sigma \times \dots \times \Sigma}_{g \text{ times}} \Big/ S_g$$

symmetric group on g elmnts

unordered g-tuples of points in Σ

<u>Remark:</u> action of S_g on $\Sigma \times \dots \times \Sigma$ is not free

<u>However:</u> $\mathrm{Sym}^g(\Sigma)$ is a smooth manifold!

Idea:

ordered g-tuples of pts in $\mathbb{C}$ $\longleftrightarrow$ unordered g-tuples of points in $\mathbb{C}$

$$z^g + a_g z^{g-1} + \dots + a_2 z + a_1 = 0 \longleftrightarrow \text{roots } r_1, \dots, r_g$$

Half-dimensional subspaces

$$\left.\begin{aligned} \mathbb{T}_\alpha &= \alpha_1 \times \dots \times \alpha_g \\ \mathbb{T}_\beta &= \beta_1 \times \dots \times \beta_g \end{aligned}\right\} \subset \mathrm{Sym}^g(\Sigma)$$

$$\mathbb{T}_\alpha \cap \mathbb{T}_\beta \subset \mathrm{Sym}^g(\Sigma)$$

Heegaard Floer generators are exactly $\mathbb{T}_\alpha \cap \mathbb{T}_\beta$

g-tuple of intersection pts b/t the α's and β's using each curve exactly once