Also consider $V_w := w \times \mathrm{Sym}^{g-1}(\Sigma) \subset \mathrm{Sym}^{g}(\Sigma)$ unordered g-tuples of pts in Σ s.t. at least one pt is w.

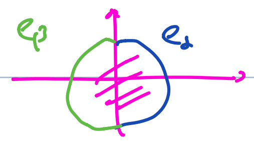
<u>Exercise</u>: Hom's lecture notes' exercises (on arXiv)

Heegaard-Floer differential.

$\widehat{CF}(H) = \langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle_{\mathbb{F}}$ $\quad \mathbb{F} = \mathbb{Z}/2$.

$\partial : \widehat{CF}(H) \to \widehat{CF}(H)$ counts holo^c disks.

Consider $\varphi : D \to \mathrm{Sym}^{g}(\Sigma)$, $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

$D =$ 

want $\varphi(-i) = x$, $\varphi(i) = y$, $\varphi(e_\alpha) \subset \mathbb{T}_\alpha$, $\varphi(e_\beta) = \mathbb{T}_\beta$.

D is a Whitney disk from x to y.

Define $\pi_2(x, y)$ = set of homotopy classes of Whitney disks from x to y.

We can picture image of φ via its 'shadow' in Σ. $\mathrm{Sym}^{g}(\Sigma) \xrightarrow{\pi} \Sigma$.

Given $x, y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, $x = (x_1, \dots, x_g)$, $y = (y_1, \dots, y_g)$

Define $\varepsilon(x, y) \in H_1(Y, \mathbb{Z})$ as follows.

Choose $a \overset{\text{arcs}}{\in} \alpha$'s, $b \in \beta$'s, s.t. $\partial a = \sum y_i - \sum x_i$, $\partial b = \sum x_i - \sum y_i$.

Then $a + b$ is a 1-cycle in Σ, so $\varepsilon(x, y) = [a + b]$ is well-defined.

<u>Exercise</u> If $\varepsilon(x, y) \neq 0$, then $\pi_2(x, y) = \emptyset$.

Technical details:

Choose a complex structure on $\Sigma \Rightarrow$ complex structure on $\mathrm{Sym}^{g}(\Sigma)$

Given $\varphi \in \pi_2(x, y)$. $\mathcal{M}(\varphi)$ be moduli space of holo^c rep^s of φ

$\mu(\varphi) :=$ expected dim of $\mathcal{M}(\varphi)$

$\mathbb{R}$-action on $\mathcal{M}(\varphi)$ coming from complex auto^s of D fixing $\pm i$.

one way to see this: use Riemann mapping thm to map disk to ∞ strip.

$\mathbb{R}$-action is translation vertically. $\hat{\mathcal{M}}(\varphi) = \mathcal{M}(\varphi)/\mathbb{R}$

$n_w(\varphi)$ = algebraic intersection # between $\varphi(D)$ and V_w

HF diff$^{=}$ $\hat{\partial}x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi)=1 \\ n_w(\varphi)=0}} \# \hat{\mathcal{M}}(\varphi) \, y$

Remark: ① $\widehat{CF}(H)$ is relatively graded: $gr(x) - gr(y) = \mu(\varphi) - 2n_w(\varphi)$, $\varphi \in \pi_2(x,y)$

Can lift to absolute grading via normalization. $\widehat{HF}(S^3) = \mathbb{F}_{(0)}$ and Cobordism maps.

② $\hat{\partial}^2 = 0$. Idea: 0-dim moduli space $\hat{\mathcal{M}}(\varphi)$ appears as the boundary of a compact 1-dim moduli space. (even # of pts)

③ Since $\pi_2(x,y) = \phi$ if $\varepsilon(x,y) \neq 0$, $(\widehat{CF}(H), \hat{\partial})$ splits as a direct sum, this is the splitting along spin^c structure

Exercise Y is $\mathbb{Q}HS^3$, $\Rightarrow$ $\dim \widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$

Def if Y is $\mathbb{Q}HS^3$, and is '=', then Y is L-space.

$CF(H) = \langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle_{\mathbb{F}[u]}$ $\quad |u| = -2$

$\partial x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi)=1}} \# \hat{\mathcal{M}}(\varphi) \, u^{n_w(\varphi)} \cdot y$. this gives $HF^-(H)$

To show HF^-, $\widehat{HF}$ are invariants of 3-manifold, need to show that Hee^2 moves induce chain homotopy equivalences & independence of choice of complex structures.

$\widehat{CF}$ is obtained from CF^- by letting $u=0$.

There is ses $0 \to CF^-(Y) \to CF^\infty(Y) \to CF^+(Y) \to 0$

$$CF^\infty(Y) \cong CF^-(Y) \otimes_{\mathbb{F}[u]} \mathbb{F}[u, u^{-1}]$$

Thm if Y is $\mathbb{Q}HS^3$, then $HF^\infty(Y, \underset{\text{spin}^c\text{-structure}}{\underbrace{s}}) = \mathbb{F}[u, u^{-1}]$

Consequence: Y $\mathbb{Q}HS^3$, then $HF^-(Y,s) \cong \mathbb{F}_{(d)}[u] \oplus \bigoplus_{i=1}^{N} \mathbb{F}_{(i)}[u]/u^{n_i}$

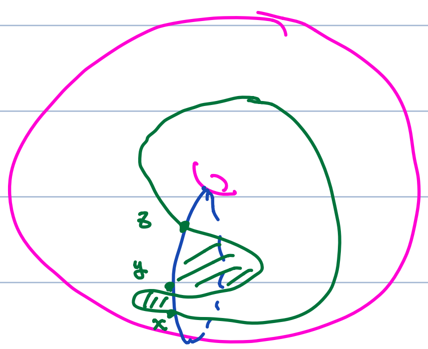
Def $d(Y,s) = \max\{gr(x) \mid x \in HF^-(Y,s),\ u^n x \neq 0 \ \forall n \geq 0\}$ (d-invariant)

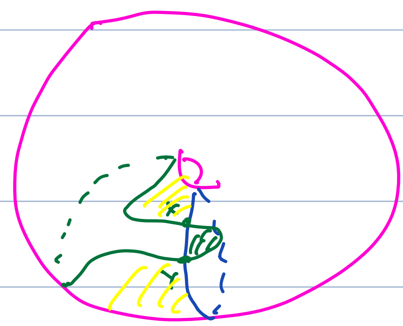
Ex $HF^-(S^3) = \mathbb{F}_{(0)}[u]$ other choice is $\mathbb{F}_{(-2)}[u]$.

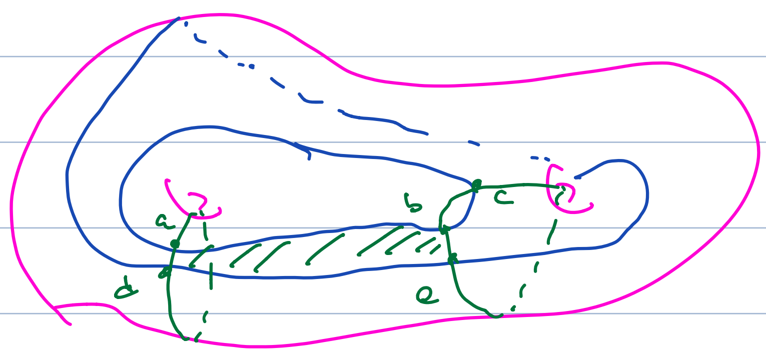
$HF^-(S^3_{+1}(T_{2,3})) = \mathbb{F}_{(-2)}[u]$. $d = -2$

Pictures from L20:

Ex







Generators: $x = \{a, e\}$, $y = \{b, d\}$, $z = \{c, d\}$

whitney disk: z to y

x to y

L21 Mar 26, 2025

HF is a version of TQFT in the following sense:

A cobordism $W: Y_0 \to Y_1$ induces an $\mathbb{F}[u]$-module homomorphism

$$HF^-(Y_0) \to HF^-(Y_1)$$ (Similarly, there are maps involving HF^+, HF^∞, $\widehat{HF}$)

Remark: Can give a $spin^c$-refinement of this.

Recall that a compact 4-manifold W is negative-definite if its intersection form is so

For simplicity, let Y be $\mathbb{Z}HS^3$.

Thm (Ozsváth-Szabó) If $W: Y_0 \to Y_1$ is negative definite cobordism, then W induce an isomorphism on HF^∞.

Denote $d_0 = \underset{\substack{\uparrow \\ \text{d-invariant}}}{d}(Y_0)$, $d_1 = d(Y_1)$,

$\mathbb{F}_{(d_0)}[u] \to \mathbb{F}_{(d_1)}[u]$, $1 \mapsto u^n$ for some n. this theorem tells us $d_0 \le d_1$.

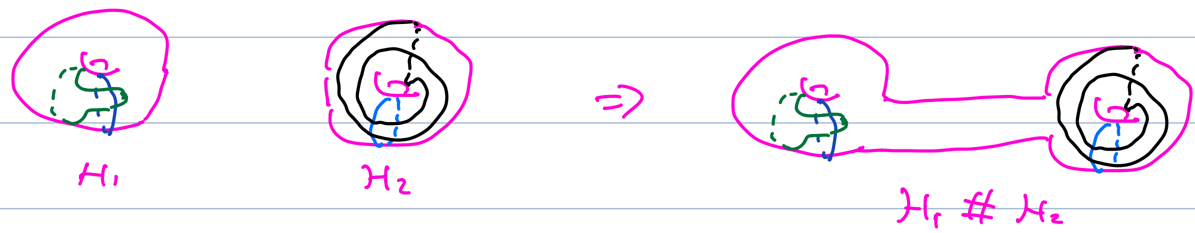
If Y_0, Y_1 are homology cobordant, $d_0 = d_1$

Connect sums:

$\left.\begin{matrix} H_1 \\ H_2 \end{matrix}\right|$ Heegaard diagram for $\left|\begin{matrix} Y_1 \\ Y_2. \end{matrix}\right.$

$H_1 \# H_2$ is a Heegaard diagram for $Y_1 \# Y_2$.

Ex



Prop $\widehat{CF}(\mathcal{H}_1 \# \mathcal{H}_2) \simeq \widehat{CF}(\mathcal{H}_1) \otimes_{\mathbb{F}} \widehat{CF}(\mathcal{H}_2)$

Pf Take connect sum near base point, Consider generator & boundary map. //

Analogous result holds for CF^-, but $CF^-(\mathcal{H}_1 \# \mathcal{H}_2) \simeq CF^-(\mathcal{H}_1) \otimes_{\mathbb{F}[u]} CF^-(\mathcal{H}_2)$

Hence $\widehat{HF}(Y_1 \# Y_2) = \widehat{HF}(Y_1) \otimes_{\mathbb{F}} \widehat{HF}(Y_2)$

$HF^-(Y_1 \# Y_2) = H_*(CF^-(\mathcal{H}_1) \otimes_{\mathbb{F}[u]} CF^-(\mathcal{H}_2))$

Remark: $\mathbb{F}[u]$ is not a field, so $\otimes$ and H_* do not commute

Ex $HF^-(\Sigma(2,3,7)) = \mathbb{F}_{(0)}[u] \oplus \mathbb{F}$

$HF^-(\Sigma(2,3,7) \# \Sigma(2,3,7)) = \mathbb{F}_{(0)}[u] \oplus \mathbb{F}_0^3 \oplus \mathbb{F}_{(-1)}$

Künneth formula over PID : Tor term

Exercise : $d(Y_1 \# Y_2) = d(Y_1) + d(Y_2)$

Fact: Y $\mathbb{Z}HS^3 \Rightarrow d(Y) \in 2\mathbb{Z}$

Ex $d(\Sigma(2,3,5)) = -2$

Hence : $d: \Theta^3_{\mathbb{Z}} \to 2\mathbb{Z}$ is surjective homomorphism.

Knot Floer homology:

doubly pointed Heegaard diagram $\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$

$\widehat{CFK}(S^3, K)$ is filtered chain complex

$\widehat{CFK}(\mathcal{H}) = \langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle_{\mathbb{F}}$

Relative Alexander grading on generators :

$A(x) - A(y) = n_z(\varphi) - n_w(\varphi)$, $\varphi \in \pi_2(x,y)$

In $g\widehat{CFK}(\mathcal{H})$, $\partial a = \partial b = \partial c = 0$

Maslov grading: $M(b) - M(c) = 1$, $M(b) - M(a) = 1 - 2 = -1$

$\uparrow$ recall that $M(x) - M(y) = \mu(\varphi) - 2n_w(\varphi)$

Absolute grading: $\widehat{HF}(S^3) = \mathbb{F}_{(0)} \Rightarrow M(a) = 0$

want to understand homology:

$\widehat{HFK}(\mathcal{H}) = H_*(g\widehat{CFK}(\mathcal{H}))$

$\Delta_{T_{2,3}}(t) = t^{-1} - 1 + t$

Ozsvath-Szabo τ-invariant:

$\tau(K) := \min\{s \mid \iota: F_s(\widehat{CFK}(\mathcal{H})) \to \widehat{CFK}(\mathcal{H})$ induces surjection on $H_*\}$

Ex $\tau(T_{2,3}) = 1$

$$\begin{array}{ccccccccccc} F_{-2} & \subseteq & F_{-1} & \subseteq & F_0 & \subseteq & F_1 & = & F_2 & = & \cdots \\ \| & & \| & & \| & & \| & & & & \\ \emptyset & & \langle c \rangle & & \langle b, c \rangle & & \langle a, b, c \rangle & & & & \\ & & & & \partial b = c & & \partial b = c & & & & \end{array}$$

Thm: (Ozsvath-Szabo) $|\tau(K)| \leq g_4^{\text{smooth}}(K)$

Connect sum:

$\widehat{CFK}(K_1 \# K_2) \cong \widehat{CFK}(K_1) \otimes_{\mathbb{F}} \widehat{CFK}(K_2)$

Consequence: $\tau(K_1 \# K_2) = \tau(K_1) + \tau(K_2)$

Here $\tau: \mathcal{C} \to \mathbb{Z}$ is surjective homomorphism

Exercise: If $\mathcal{H}_1, \mathcal{H}_2$ doubly pointed Heegaard diagrams for K_1, K_2, find a doubly pointed Heegaard diagram for $K_1 \# K_2$

$$\hat{\partial} x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi)=1 \\ n_w(\varphi)=0}} \# \hat{\mu}(\varphi)\, y$$

Alexander grading on generators induces Alexander filtration on $\widehat{CFK}(H)$:

$A(\sum x_i) = \max\{A(x_i)\}$, $A(x) \geq A(\hat{\partial} x)$

Filtration: $F_s(\widehat{CFK}(H)) = \langle x \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta \mid A(x) \leq s \rangle_{\mathbb{F}_2}$, $\cdots \subseteq F_{s-1} \subseteq F_s \subseteq \cdots$

$g\widehat{CFK}(H)$ = associated graded complex.

$$= \bigoplus_s F_s(\widehat{CFK}(H)) / F_{s-1}(\widehat{CFK}(H)).$$

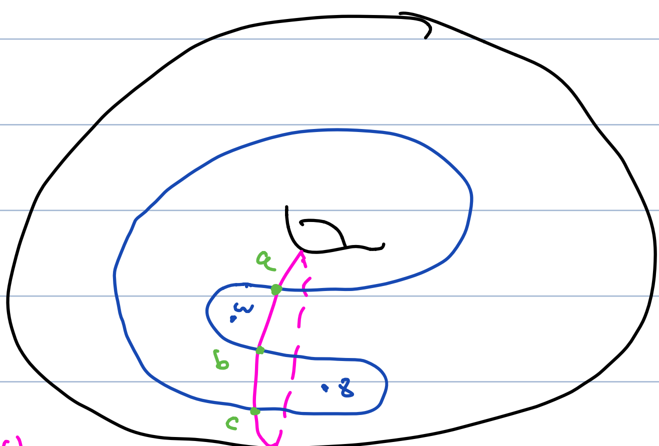
$\hat{\partial}_g : g\widehat{CFK}(H) \to g\widehat{CFK}(H)$.

$$\hat{\partial}_g(x) = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi)=1 \\ n_w(\varphi)=0=n_z(\varphi)}} \# \hat{\mu}(\varphi)\, y$$

i.e. we only care the part of $\hat{\partial}$ that preserves Alexander grading.

$\widehat{HFK}(K) = H_*(g\widehat{CFK}(H))$.

Ex



Convention $\pi_2(x,y)$

$\widehat{CFK}(H)$

$\partial a = 0 = \partial c$, $\partial b = c$

$A(b) - A(c) = 1$, $A(b) - A(a) = -1$

$A(a) = 1$

$A(b) = 0$

$A(c) = -1$

More flavor of Knot Floer homology:

$R = \mathbb{F}[u, v]$, bigrading $gr = (gr_u, gr_v)$

$gr_u = (-2, 0)$, $gr_v = (0, -2)$

$CFK_R(H) = \langle \mathbb{T}_\alpha \cap \mathbb{T}_\beta \rangle_R$

$$\partial_R x = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\varphi \in \pi_2(x,y) \\ \mu(\varphi) = 1}} \# \hat{\mathcal{M}}(\varphi) \, u^{n_w(\varphi)} v^{n_z(\varphi)} y$$