8803

Week 13

Wednesday pg 2

<u>Last time:</u>

Alternate formulation of knot Floer homology

$R = \mathbb{F}[U,V]$ bigraded ring $\quad gr = (gr_U, gr_V)$

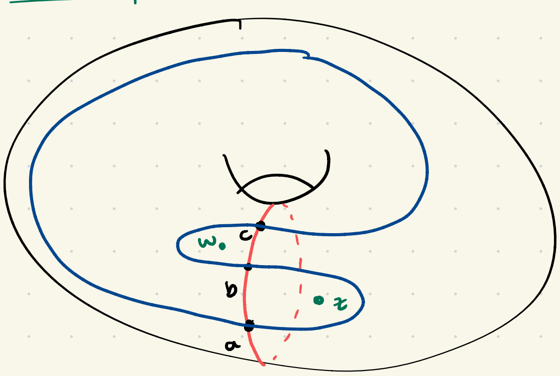
gr_U = Maslov grading

$gr(U) = (-2,0)$

$gr(V) = (0,-2)$

$$dx = \sum_{y \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta} \sum_{\substack{\phi \in \pi_2(x,y) \\ \mu(\phi)=1}} \# \widehat{\mathcal{M}}(\phi)\, U^{n_w(\phi)} V^{n_z(\phi)}\, y$$

<u>Example:</u>



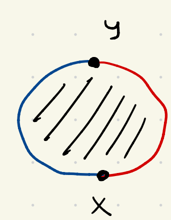
$\phi \in \pi_2(x,y)$

how to get relative Maslov grading

$gr_U(x) - gr_U(y) = \mu(\phi) - 2n_w(\phi)$

$\underbrace{gr_V(x) - gr_V(y)}_{\text{how to get relative z grading}} = \mu(\phi) - 2n_z(\phi)$

$$A(x) - A(y) = n_z(\phi) - n_w(\phi)$$
$$= \frac{1}{2}\Big(gr_U(x) - gr_U(y) - \big(gr_V(x) - gr_V(y)\big)\Big)$$



$$\left.\begin{aligned} \partial a &= 0 \\ \partial b &= Vc + Ua \\ \partial c &= 0 \end{aligned}\right\}$$ can see that's everything by passing thru. univ. cover

Convenient way to depict:

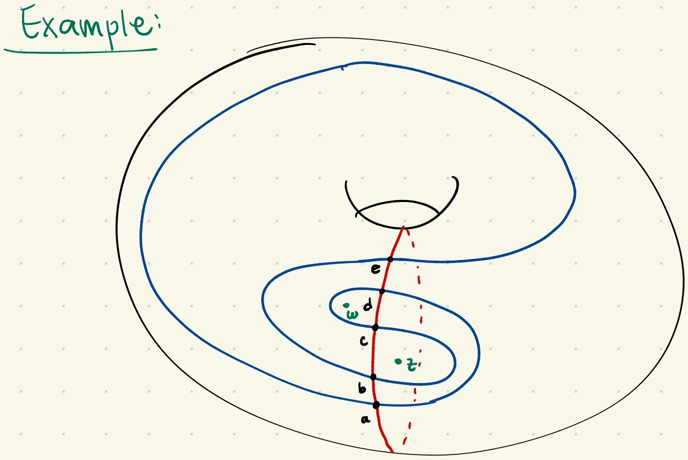
$$\begin{tikzcd} Ua & b \arrow[l] \arrow[d] \\ & Vc \end{tikzcd}$$

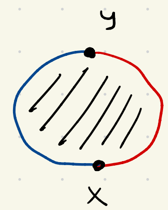
$$\begin{array}{cccc} \cdots & u^2 & u & 1 \\ & u^2v & uv & v \\ \cdots & & uv^2 & v^2 \\ & & & \vdots \end{array}$$

Can check the following gradings of the generators:

	gr_u	gr_v	Alex.
a	0	-2	1
b	-1	-1	0
c	2	0	-1

Example:





Looking for disks where the corners are acute.

$$\partial a = 0$$
$$\partial b = Va + Ve$$
$$\partial c = Ub + Vd$$
$$\partial d = Ua + Ue$$
$$\partial e = 0$$

$$\begin{tikzcd} Ub \arrow[d] \arrow[dr] & & c \arrow[ll] \arrow[dd] \\ & UVe & \\ UVa & & Vd \arrow[ll] \arrow[ul] \end{tikzcd}$$

Often convenient to do a change of basis.

$$\begin{tikzcd} & Ub \arrow[d] & c \arrow[l] \arrow[d] \\ UVa & UV(a+e) & Vd \arrow[l] \end{tikzcd}$$

Now the differential is simplified.

$$\begin{tikzcd} Ub \arrow[d] & c \arrow[l] \arrow[d] \\ UV(a+e) & Vd \arrow[l] \end{tikzcd} \qquad UVa \qquad\qquad \begin{tikzcd} Ua & b \arrow[l] \arrow[d] \\ & Vc \end{tikzcd}$$

symmetry.

Says something about chain homotopy equiv.

$\mathcal{H} = (\Sigma, \alpha, \beta, w, z)$ for $K \subset S^3$

this invariant is not sensitive to string orientation

the reverse!

Then $(-\Sigma, \underbrace{\beta, \alpha}, \underbrace{w, z})$ describes K^r

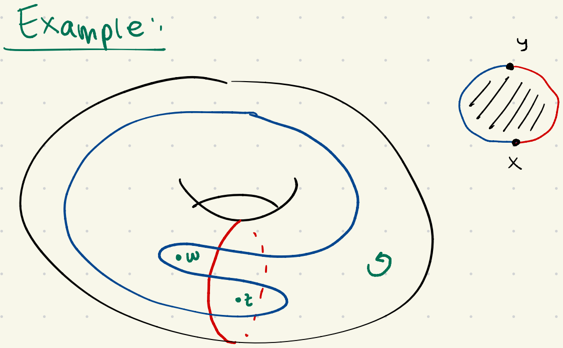
3-mfd unchanged but now reverse of knot

reverses 3-mfd

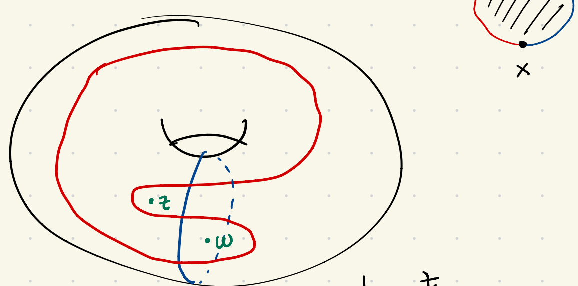
reverses 3-mfd

So $(-\Sigma, \beta, \alpha, z, w)$ describes K.

Example:



$(-\Sigma, \beta, \alpha, z, w)$



$b \xrightarrow{z} a$

These both describe $K \subset S^3$

Compare $CFK_R(\mathcal{H})$ $\qquad$ $CFK_R(\mathcal{H}')$

same generators

same differential, but with roles of $\mathcal{U}$ and V swapped.

$\Longrightarrow$ Swapping $\mathcal{U}$ and V (and also gr_U, gr_V) results in chain homotopy equivalence.

$CFK_R(K)$ and concordance

Theorem (Zemke)

If K_0 and K_1 are concordant, then $\exists$ absolutely gr_U, absolutely gr_V R-equivariant maps

$$\begin{tikzcd} CFK_R(K_0) \arrow[r, bend left, "f"] & CFK_R(K_1) \arrow[l, bend left, "g"] \end{tikzcd}$$

such that f and g induce isomorphisms on

$$H_*\left(CFK_R(K_i)/U\right) \Big/ V\text{-torsion}$$

free fin. gen. chain cpx over U

<u>Observe</u> $\underbrace{CFK_R(K)/U}_{}$

free fin. gen. $\mathbb{F}[V]$-module (now a P.I.D)

<u>Fact:</u> $H_*\left(CFK_R(K)/U\right) \cong \mathbb{F}[V] \oplus \bigoplus_{i=1}^{N} \mathbb{F}[V]/V^{n_i}$

one free part (single spin c structure)

<u>Recall:</u> $H_*\left(CFK_R(K)/U\right) \cong H_*\left(CFK_R(K)/V\right)$

$gr_U \longleftrightarrow gr_V$

$R = \mathbb{F}[U,V]$

one variable for each basept, counting how many times they're crossed.

$HFK^-(K) := H_*\left(CFK_R(K)/V\right)$ (up to swapping U & V)

fin. gen. gr $\mathbb{F}[U]$-module

Proposition

$$\tau(K) = -\frac{1}{2} \max \{ gr_v(x) \mid x \in H_*(CFK_R(K)/u),\ V^n x \neq 0\ \forall n > 0 \}$$

Q: How can we compute things?

- computer program that computes $\widehat{HFK}$, τ, ε, ν, ...

  bigraded theory

  haven't talked about

  strictly more info. homology of graded cpx

  (Szabó's website, Snappy)

  uses some alt. defn of Knot Floer homology, taking slices of Platt closure ??

- alternating knots: $CFK_R(K)$ determined by $\Delta_K(t)$ and $\sigma(K)$

  bad: not telling us anything new b/c classical invariants

  good: can actually compute

Defn: A knot $K \subset S^3$ is called an L-space knot if $\exists$ a $r > 0$ s.t. $S^3_r(K)$ is an L-space.

Recall: A $\mathbb{Q}HS^3$ Y is an L-space if $\dim \widehat{HF}(Y) = |H_1(Y;\mathbb{Z})|$

In general, $\dim \widehat{HF}(Y) \geq |H_1(Y;\mathbb{Z})|$

- L-space knots: $CFK_R(K)$ is completely determined by $\Delta_K(t)$

- Linear combinations of the above:

Künneth formula:

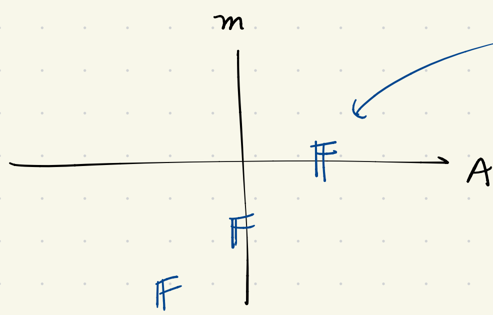
$$CFK_{\mathcal{R}}(K_1 \# K_2) \simeq CFK_{\mathcal{R}}(K_1) \otimes_{\mathcal{R}} CFK_{\mathcal{R}}(K_2)$$

Alternating Knots

Theorem: (Osváth-Szabó)

Let K be an alternating knot. Then $\widehat{HFK}(K)$ is supported in a single diagonal of slope 1 in the Alexander-Maslov gr plane and $\tau(K) = \frac{-1}{2}\sigma(K)$

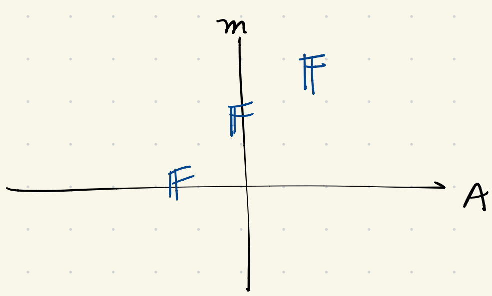
Example: $K = T_{2,3}$



comes from Heeg. Floer hom of S^3 being supported in Maslov grading 0

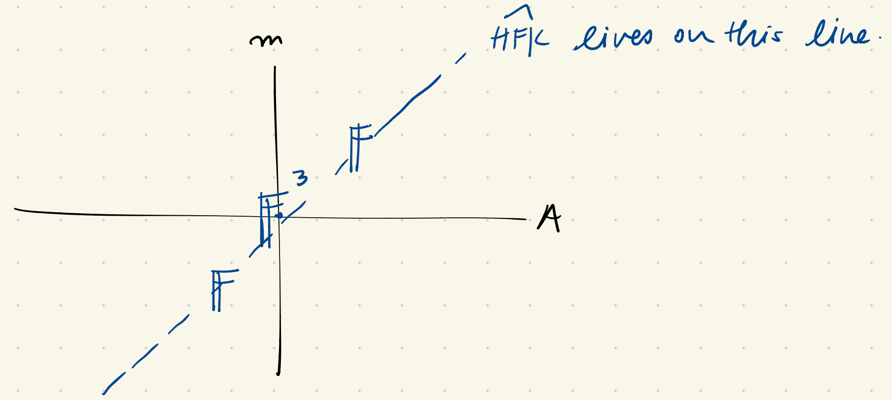
$$\Delta_K(t) = t^{-1} - 1 + t$$
$$\sigma(K) = -2$$

Example: $K = -T_{2,3}$



$$\Delta_K(t) = t^{-1} - 1 + t$$
$$\sigma(K) = 2$$

Example: $K = 4_1$



$$\Delta_K(t) = -t^{-1} + 3 - t$$

$$\sigma(K) = 0$$

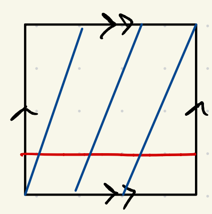
Exercise: For K alternating, $CFK_R(K)$ is also completely determined by $\Delta_K(t)$ and $\sigma(K)$

L-space knots

Examples:

$$\dim \widehat{HF}(L(p,q)) = p$$

lens spaces are L-spaces



$L(3,1)$

Examples:

Exercise: $S^3_{pq \pm 1}(T_{p,q})$ is a lens space.

Hence positive torus knots are L-space knots

Theorem (Ozsváth-Szabó)

If K is an L-space knot, then $S^3_r(K)$ is an L-space for all $r > 2g-1$

Theorem (Ozsváth-Szabó)

If K is an L-space knot, then $\tau(K) = g(K)$ and $\widehat{HFK}(K)$ is at most 1-dimensional in each Alexander grading.

Corollary

If K is an L-space knot, then

- K is fibered, and
- non-zero coefficients of $\Delta_K(t)$ are ± 1

Big Q: Which knots in S^3 admit surgeries to lens spaces?

A consequence of this result is the following recipe for determining the knot Floer complex $CFK_R(K)$ from Alexander polynomial $\Delta_K(t)$ for K an L-space knot:

If $\Delta_K(t) = t^{n_0} - t^{n_1} + t^{n_2} - t^{n_3} + \dots + t^{n_m}$

then $CFK_R(K)$ has $|\Delta_K(-1)|$

$x_0, x_1, x_2, \dots, x_m$ (generators)

and

$$\partial x_1 = U^{n_0 - n_1} x_0 + V^{n_1 - n_2} x_2 \qquad \text{(differentials)}$$

$$\partial x_3 = U^{n_2 - n_1} x_2 + V^{n_3 - n_4} x_4$$

$$\vdots$$

Example:

$K = T_{4,5}$

Recall: $\Delta_{T_{p,q}}(t) = \dfrac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$

$$\Delta_{T_{4,5}}(t) = t^6 - t^5 + t^2 - 1 + t^{-2} - t^{-5} + t^6$$

$CFK_R(K)$ has 7 generators: $x_0, x_1, \dots, x_6$ and

$$\partial x_1 = U x_0 + V^3 x_2$$

$$\partial x_3 = U^2 x_2 + V^2 x_4$$

$$\partial x_5 = U^3 x_4 + V x_6$$