8803

Week 14

Monday pg 2

Wednesday pg 11

Last time:

K is an L-space knot if $\exists$ r s.t. $S^3_r(K)$ is an L-space.

Ex: positive torus knots are L-space knots

If K is an L-space knot, then $CFK_R(K)$ is determined by $\Delta_K(t)$

Ex: $K = T_{4,5}$

$$\Delta_K(t) = t^6 - t^5 + t^2 - 1 + t^{-2} - t^{-5} + t^{-6}$$

each term in alex poly is a generator

generators: $x_0, x_1, \dots, x_6$

$$\partial x_0 = \partial x_2 = \partial x_4 = \partial x_6$$

$$\partial x_1 = U x_0 + V^3 x_2$$

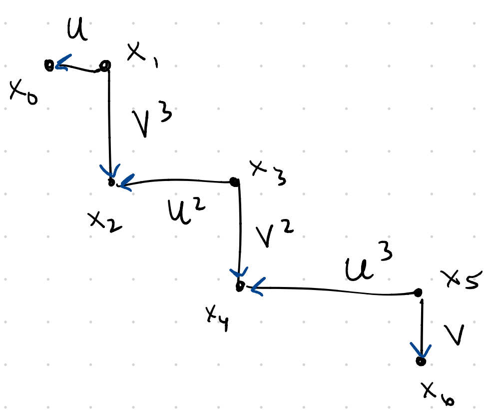
U^1 1 = difference of 6 and 5

V^3 3 = difference of 5 and 2

$$\partial x_3 = U^2 x_2 + V^2 x_4$$

$$\partial x_5 = U^3 x_4 + V x_6$$

$\Rightarrow$ U, V exponents given by difference in exponents in $\Delta_K(t)$



More concordance homomorphisms

Theorem (Dai–H–Stoffregen–Truong)

For each $k \in \mathbb{Z}_{>0}$, there exists $\varphi_k : \mathcal{C} \to \mathbb{Z}$

Moreover,
$$\bigoplus_k \varphi_k : \mathcal{C} \longrightarrow \mathbb{Z}^\infty \quad \text{is surjective.}$$

Q: How are the φ_k defined?

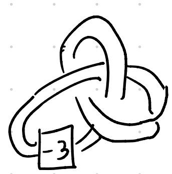
Ex: $K = T_{4,5}$

Can record the path x_0 to x_6 from above by $(1, -3, 2, -2, 3, -1)$ which are the signed exponents in path from x_0 to x_6

exponents on V in same dir

exponents on U w/ oppossing direction of path

Ex: $K = T_{2,3;2,1}$ ← 2,1 cable of trefoil



$\partial x_0 = 0$

$\partial x_1 = U x_0 + V^2 x_2 + UV x_4$

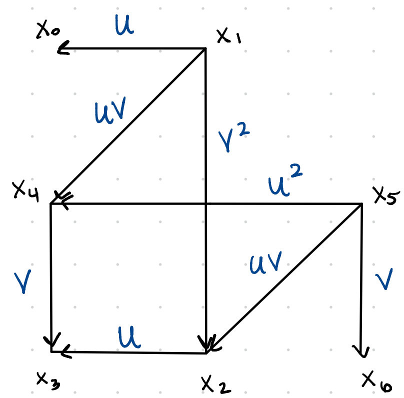
$\partial x_2 = U x_3$

$\partial x_3 = 0$

$\partial x_4 = V x_3$

$\partial x_5 = U^2 x_4 + UV x_6 + V x_6$

$\partial x_6 = 0$



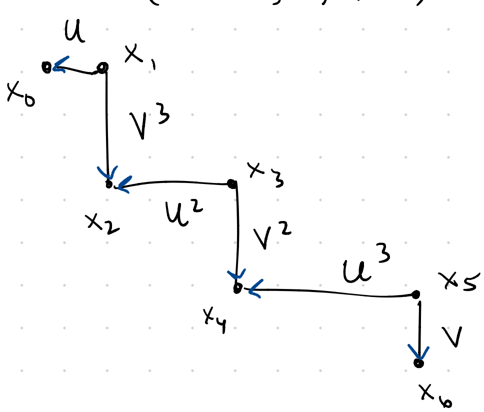
complex

still have path alternating U's and V's

$(1, -2, -1, 1, 2, -1)$ sequence

Q: From a sequence, can we get a chain complex?

Example: $(1,-3,2,-2,3,-1)$



given sequence, get generators

$$x_0 \xleftarrow{u} x_1 \xrightarrow{v^3} x_2 \xleftarrow{u^2} x_3 \xrightarrow{v^2} x_4 \xleftarrow{u^3} x_5 \xrightarrow{v} x_6$$

Example: $(1,-2,-1,1,2,-1)$

$$x_0 \xleftarrow{u} x_1 \xrightarrow{v^2} x_2 \xrightarrow{u} x_3 \xleftarrow{v} x_4 \xleftarrow{u^2} x_5 \xrightarrow{v} x_6$$

$$\partial^2 \neq 0$$

$$\partial^2 x_1 = \partial(u x_0 + v^2 x_2) = v^2 \partial x_2 = u v^2 x_3$$

The fix is to set $uv = 0$

For C a chain complex over $\mathbb{F}[u,v]/uv$, let ∂_u be the induced boundary map on C/v and let ∂_v be the induced boundary map on C/u.

Defn: Given a sequence $(a_i)_{i=1}^{2N}$, $a_i \in \mathbb{Z} \setminus \{0\}$, the associated standard complex has generators (over $\mathbb{F}[u,v]/uv$)

$$x_0, x_1, \ldots, x_{2N}$$

1. For i odd

   a. if $a_i > 0$, then $\partial_U x_i = U^{a_i} x_{i-1}$

   b. if $a_i < 0$, then $\partial_U x_{i-1} = U^{|a_i|} x_i$

2. For i even,

   a. if $a_i > 0$, then $\partial_V x_i = V^{a_i} x_{i-1}$

   b. If $a_i < 0$, then $\partial_V x_i = V^{|a_i|} x_i$

Example:

$(-1, 3, 1, -1, -3, 1)$

$$\begin{tikzcd} x_0 \arrow[r, "U"] & x_1 & x_2 \arrow[l, "V^3"'] & x_3 \arrow[l, "U"'] \arrow[r, "V"] & x_4 \arrow[r, "U^3"] & x_5 & x_6 \arrow[l, "V"'] \end{tikzcd}$$

$$\partial x_3 = U x_2 + V x_4$$

$\boxed{\text{Theorem}}$ (DHST)

Every knot Floer complex over $\mathbb{F}[U,V]/UV$ has a standard complex as a direct summand, and it is unique (up to chain homotopy equivalence). Furthermore, this standard complex is a concordance invariant.

$K \subset S^3$

$K \subset S^3 \rightsquigarrow CFK_{\mathbb{F}[U,V]}(K) \rightsquigarrow CFK_{\mathbb{F}[U,V]/UV}(K) \rightsquigarrow$ standard complex

Upshot: $K \rightsquigarrow$ sequence

well-defined map of sets $\quad \mathcal{C} \longrightarrow \{\text{sequences}\}$

$$[K] \longmapsto \text{standard complex sequence}$$

Recall: standard complexes are in bijection with finite sequences (a_i) $\quad a_i \in \mathbb{Z} \setminus \{0\}$

Q: Can we put a binary operation on the set of such sequences to obtain a group homomorphism

$$\mathcal{C} \longrightarrow \{\text{sequences}\} \text{ ?}$$

A: sort of ...

Ex: $C_1 = (1,-1)$

$$\begin{tikzcd} x_0 & x_1 \arrow[l, "U"'] \arrow[r, "V"] & x_2 \end{tikzcd}$$

$C_2 = (1,-1)$

$$\begin{tikzcd} y_0 & y_1 \arrow[l, "U"'] \arrow[r, "V"] & y_2 \end{tikzcd}$$

$C_1 \otimes_{\mathbb{F}[U,V]/UV} C_2$

$$\begin{tikzcd}
x_0y_2 & x_1y_2 \arrow[l, "U"'] \arrow[r, "V"] & x_2y_2 \\
x_0y_1 \arrow[u, "V"'] \arrow[d, "U"] & x_1y_1 \arrow[l, "U"'] \arrow[r, "V"] \arrow[u, "V"'] \arrow[d, "U"] & x_2y_1 \arrow[u, "V"'] \arrow[d, "U"] \\
x_0y_0 & x_1y_0 \arrow[l, "U"'] \arrow[r, "V"] & x_2y_0
\end{tikzcd}$$

change of basis:

$$\begin{tikzcd}
x_0y_2 + x_2y_0 & x_1y_2 + x_2y_1 \arrow[l, "u"'] & x_2y_2 \\
x_0y_1 + x_1y_0 \arrow[u, "v"'] & x_1y_1 \arrow[l, "u"] \arrow[u, "v"'] & x_2y_1 \arrow[u, "v"'] \arrow[d, "u"] \\
x_0y_0 & x_1y_0 \arrow[l, "u"] \arrow[r, "v"'] & x_2y_0
\end{tikzcd}$$

Spot the standard complex summand!

$(1,-1,1,-1)$

* Not as simple as concatenation — recall tensor prod. is commutative.

Given two sequences, (a_i) and (b_i) can take their associated standard complexes, tensor them together, and then perform a change of basis to obtain a standard complex summand.

Examples:

$\varphi_j = \begin{cases} 1 & i=1 \\ 0 & \text{otherwise} \end{cases}$

1. $(1,-1) \otimes (1,-1) = (1,-1,1,-1)$

2. $(1,-1) \otimes (1,-2,2,-1) = (1,-2,1,-1,2,-1)$

3. $(1,-1) \otimes (2,-2) = (1,-1,2,1,-1,-2,1,-1)$ $\qquad \varphi_j = \begin{cases} 1 & j=1 \\ 1 & j=2 \\ 0 & j \geq 3 \end{cases}$

$\varphi_j = \begin{cases} 1 & j=1 \\ 0 & \text{o/w} \end{cases}$ $\qquad \varphi_j = \begin{cases} 1 & j=2 \\ 0 & \text{o/w} \end{cases}$

Open Problem: give a precise description of the group operation on sequences

Nevertheless,

Def'n: given a finite sequence of nonzero integers (a_i) $a_i \in \mathbb{Z} \setminus \{0\}$
and a positive integer $j \in \mathbb{Z}_{>0}$

$$\varphi_j(a_i) = \#\{a_i = j \mid i \text{ odd}\} - \#\{a_i = -j \mid i \text{ odd}\}$$

Theorem

φ_j is a homomorphism

Example:

$$(a_1, a_2, \ldots, a_{2n}) \otimes (-a_1, -a_2, \ldots, -a_{2n}) = 0 \qquad \text{inverses}$$

Exercise:

$$\varphi_j(T_{n,n+1}) = \begin{cases} 1 & j = -1, \ldots, n-1 \\ 0 & \text{otherwise} \end{cases}$$

→ L space knots, $\Delta_K(t)$, read off complex, ...

Corollary

$\bigoplus_j \varphi_j : \mathcal{C} \longrightarrow \mathbb{Z}^\infty$ is surjective.

Consider

$$0 \longrightarrow \mathcal{C}_{TS} \longrightarrow \mathcal{C}_{smooth} \longrightarrow \mathcal{C}_{top} \longrightarrow 0$$

topologically slice knots

subgroup of concordance group

Example: $Wh(K) \in \mathcal{C}_{TS} \ \forall K$ by Freedman since $\Delta_{Wh(K)}(t) = 1$

$Wh(T_{2,3})$ is not smoothly slice (using Donaldson)

Theorem (Ozsváth–Stipsicz–Szabó)

$$\boxed{\mathcal{C}_{TS} \text{ contains a } \mathbb{Z}^\infty \text{ direct summand}}$$

Proof relies on concordance homomorphism Υ_K upsilon

Reproof of this result using φ_j

Theorem DHST

$$\boxed{\bigoplus_j \varphi_j : \mathcal{C}_{TS} \longrightarrow \mathbb{Z}^\infty \text{ surjective}}$$

Proof: Let $D = Wh(T_{2,3})$

Observe D top slice $\implies D_{n,n+1} \underset{top}{\sim} U_{n,n+1} = T_{n,n+1}$

n,n+1 cable

i.e. $D \underset{top}{\sim} U$

$$\Longrightarrow D_{n,n+1} \# -T_{n,n+1} \underset{top}{\sim} U$$

$$D_{n,n+1} \# -T_{n,n+1} \in \mathcal{C}_{TS}$$

Claim: $\varphi_j(D_{n,n+1}) = \begin{cases} n & j=1 \\ 1 & 1<j<n-1 \text{ or } j=n \\ 0 & j=n-1>1 \text{ or } j>n \end{cases}$

Claim & earlier exercise $\left(\varphi_j(T_{n,n+1})\right) \Longrightarrow$

$$\varphi_j\left(D_{n,n+1} \# -T_{n,n+1}\right) = \begin{cases} 1 & j=n \\ 0 & j>n \end{cases}$$

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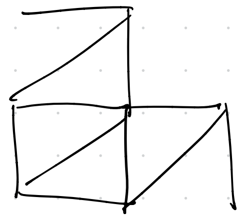
Dr Hom – No office hours today (Wednesday)

Correction from last time:

$T_{2,3;2,1}$



Last time:

$K \rightsquigarrow CFK(K) \rightsquigarrow$ standard complex $\rightsquigarrow$ sequence (a_i)

chain homot. cpx is a knot invariant but not concordance inv

is a concord. inv.

If K is an L-space knot, then (a_i) is determined by $\Delta_K(t)$

$\Leftarrow$ $\Rightarrow$

Theorem (Hedden, Hom)

longit. winding

$K_{p,q}$ = cable knot

Let $p>0$. Then $K_{p,q}$ is an L-space knot

$\Longleftrightarrow$

K is an L-space knot and $q > p(2g(K)-1)$

Proof: $(\Leftarrow)$

Consider $S^3_{pq}(K_{p,q})$

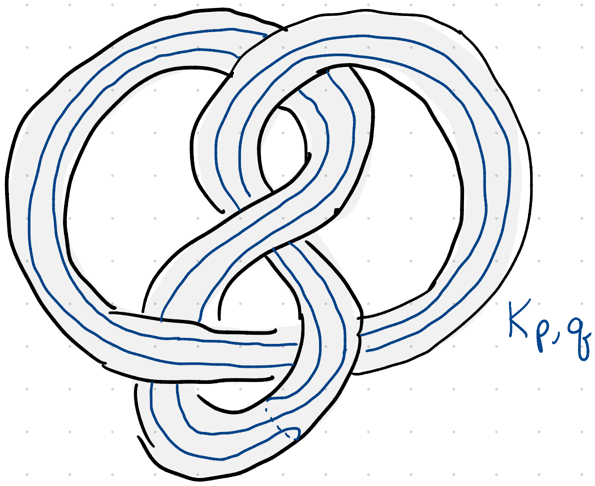
Claim: $S^3_{pq}(K_{p,q}) = S^3_{q/p}(K) \# L(p,q)$

Proof of claim: For a knot $J \subset S^3$, let $E(J) = S^3 - \nu(J)$ (Exterior)

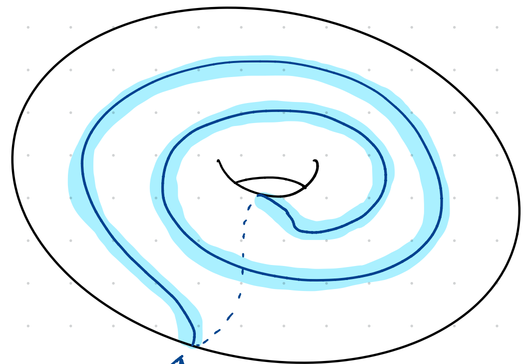
$T_J = \partial \nu(J)$

$E(K_{p,q}) = E(K) \cup_{T_K - A} \nu(K)$

where $A = \nu(K_{p,q}) \cap T_K$



$K = 4_1$

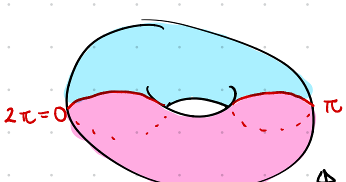


Now glue solid torus $S^1 \times D^2$ to exterior $E(K_{p,q})$ such that $\{0\} \times \partial D^2$ maps to a pq-framed longitude λ of $K_{p,q}$

Exercise: λ is the surface framing of $K_{p,q}$ on T_K

(draw push off and compute linking #)

Decompose $S^1 \times D^2$ as $([0,\pi] \times D^2) \cup ([\pi, 2\pi] \times D^2)$



think of as thickened meridinal disks

Exercise:

1) $E(K) \cup ([0,\pi] \times D^2) = S^3_{q/p}(K) - B^3$

$\underbrace{}_{}$ some surgery on $K - B^3$ and then check framing

2) $\nu(K) \cup ([\pi, 2\pi] \times D^2) = L(p,q) - B^3$

3.) $(\{0\} \times D^2) \cup (T_K - A) \cup (\{\pi\} \times D^2) = S^2$

This completes the proof of claim.

Hence $S^3_{pq}(K_{p,q}) = S^3_{q/p}(K) \# L_{p,q}$

So if K is an L-space knot, and $q/p \geq 2g(K) - 1$, then $S^3_{q/p}(K)$ is an L-space.

$L(p,q)$ is an L-space

Since connected sums of L-spaces are L-spaces (since $\widehat{HF}(Y_1 \# Y_2) = \widehat{HF}(Y_1) \otimes \widehat{HF}(Y_2)$) (Exercise: compute H_1)

It follows that $S^3_{pq}(K_{p,q})$ is an L-space, as desired.

(For $\Rightarrow$, see H. A note on cabling and L-space surgeries)

Exercise: Let $p > 0$. $T_{2,3;p,q}$ is an L-space knot $\iff$ $q > p$

Exercise: Find sequence (a_i) for $T_{2,3;n,n+1}$

Recall: $\Delta_{K_{p,q}}(t) = \Delta_K(t^p) \cdot \Delta_{T_{p,q}}(t)$

$$\Delta_{T_{p,q}}(t) = \frac{(t^{pq}-1)(t-1)}{(t^p-1)(t^q-1)}$$

Goal: $\varphi_j(D_{n,n+1})$ $\qquad D = Wh(T_{2,3})$

Proposition: (Lipschitz - Ozsváth - Thurston)

If K_0 and K_1 have the same standard complex representative, then (satellites) $P(K_0)$ and $P(K_1)$ also have the same standard complex representative.

In other words, satelliting gives a well-defined map on sequences (a_i)

Proposition (Hedden)

$D = Wh(T_{2,3})$ has the same standard complex as $T_{2,3}$ i.e. $(1,-1)$

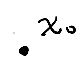
Hence $\varphi_j(D_{n,n+1}) = \varphi_j(T_{2,3;n,n+1})$

Q: Besides the φ_j, what other concordance invariants can we get from the sequences?

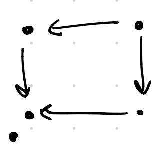
Let (a_i) be the sequence associated to K

Defn: $$\varepsilon(K) = \begin{cases} 1 & \text{if } a_1 > 0 \\ -1 & \text{if } a_1 < 0 \\ 0 & \text{otherwise} \end{cases}$$ (i.e. (a_i) is trivial sequence)

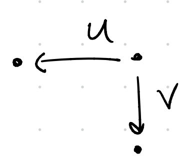
Example:

Unknot has trivial sequence 

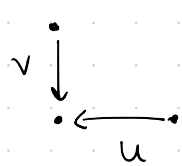
Example:

4_1  also has trivial sequence

Example:

$T_{2,3}$  $(1,-1)$ $\varepsilon(T_{2,3}) = 1$

Example:

$T_{-2,3}$  $(-1,1)$ $\varepsilon(T_{-2,3}) = -1$

Observe:

$$\varepsilon : \mathcal{C} \longrightarrow \{-1, 0, 1\}$$ not a homomorphism.

Not a group

We can still ask...

Q: How does ε behave under connected sum $\#$?

A: Like a sign

Proposition

$\varepsilon(K_1)$	$\varepsilon(K_2)$	$\varepsilon(K_1 \# K_2)$
$+1$	$+1$	$+1$
-1	-1	-1
0	± 1	± 1
-1	$+1$	anything

Observe:

$$K \rightsquigarrow CFK(K)$$

$$-K \rightsquigarrow CFK(K)^*$$

$$K_1 \# K_2 \rightsquigarrow CFK(K_1) \otimes CFK(K_2)$$

K slice $\rightsquigarrow$ $\varepsilon(CFK(K)) = 0$

Consider

some equivalence relation

$$\mathcal{CFK} := \left(\left\{ \begin{array}{c} \text{complexes over } \mathbb{F}[U,V] \\ \text{that satisfy the same} \\ \text{algebraic properties as } CFK(K) \end{array} \right\} \Big/ \sim \;,\; \otimes \right)$$

sometimes called ε-equivalence, or local equivalence over $\mathbb{F}[U,V]/UV$

$$C_0 \sim C_1 \iff \varepsilon(C_0 \otimes C_1^*) = 0$$

Upshot:

$$\mathcal{C} \longrightarrow \mathcal{CFK} \text{ group}$$
$$[K] \longmapsto [CFK(K)]$$

OPEN Q: Is this surjective?

and in fact

$$CFK(K_0) \sim CFK(K_1) \iff K_0, K_1 \text{ have the same sequence } (a_i)$$