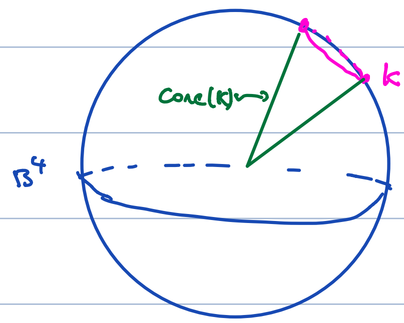
L25 14th, Apr, 2025.

[What surfaces can a knot bound in B^4?

- Smoothly slice knots bound a smoothly embedded disk in B^4.
- Top^{l} slice knots bound embedded locally flat disks in B^4.

Non-slice knots?

- Knots bound smoothly embedded, compact, orientable surfaces in B^4 (Seifert surfaces)
- Knots always bound an immersed disk. (use null homotopy of a knot).



What about embedded disks?

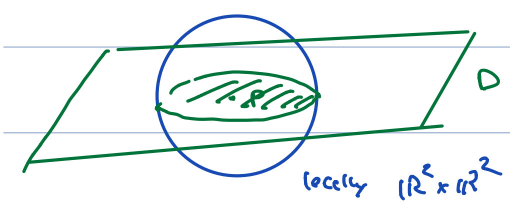
(not necessarily locally flat)

$B^4 = \text{Cone}(S^3)$, then Cone(K) will give an embedded disk.

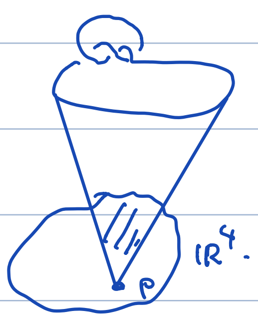
Remark Since there are knots that are not top^{l} slice, this disk need not be locally flat.

[Prop If K is non-trivial, then disk is not locally flat at cone pt.

Pf If D is locally flat at a pt p.



$$\pi_1(\mathbb{R}^4\setminus D) = \pi_1(\mathbb{R}^2\times\mathbb{R}^2\setminus\mathbb{R}^2\times\{0\})$$
$$= \pi_1(\mathbb{R}^2\times(\mathbb{R}^2\setminus\{0\}))$$
$$= \mathbb{Z}$$



$\mathbb{R}^4\setminus D$ retracts onto $B^4\setminus\text{Cone}(K) = (S^3\setminus K)\times I$.

Thus $\pi_1(\mathbb{R}^4\setminus D) \twoheadrightarrow \pi_1(S^3\setminus K)$

But $\pi_1(S^3\setminus K)$ is never $\mathbb{Z}$ if K is not unknot.

Then $\pi_1(\mathbb{R}^4\setminus D)\neq\mathbb{Z}$. $\Rightarrow$ Cone(K) is not locally flat at cone pt //

Def A disk in a 4-manifold is a PL-disk if it's smoothly embedded except at some singularities that are $\mathrm{Cone}(K) \subseteq B^4$.

We can replace a PL-disk by one with only one singularity.

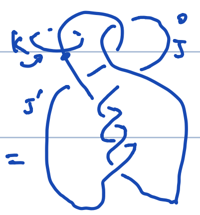
Take γ be the path between all cone pts. In nbhd of γ, see all cone singularities on $K_1, \dots, K_n$, can replace w/ $\mathrm{Cone}(K_1 \# \cdots \# K_n)$.

If $K \subseteq S^3$, and X 4-manifold s.t. $\partial X = S^3$. does K bound PL disk in X?

Yes use collar of ∂X to find B^4 containing K.

Conj $\exists$ contractible 4-manifold Z w/ knot $K \subseteq \partial Z$ s.t. K cannot bound a PL disk in Z.

Thm. The conjecture is true.

$Z =$ 

$\dot{\bigcirc}$ = remove disk the unknot bounds from B^4.

$\overset{0}{\bigcirc}{}^{J}$ = attach $D^2 \times D^2$ to B^4 along nbhd(J), so that $S^1 \times \{1\}$ goes to λ_J.

Exercise : Z is contractible

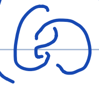
∂Z is 3-manifold obtained from 0-surgeries on both components.

Exercise Find a different 4-manifold Z' w/ same boundary as Z but K bounds smooth embedded disk in Z'.

Switch roles of • and ∘.

Modified Zeeman Conjecture: $\exists$ Y^3 which is a contractible 4-manifold and $K \subseteq Y$ which can't bound any PL disk in any contractible 4-manifold w/ boundary?

Thm This conjecture is true.

Pf $Y = S^3_{-\frac{1}{2}}($  $)$, K = core curve of solid torus used in surgery.

Fact Y bounds contractible.

We'll show K can't bound PL-disk in any $\mathbb{Z}HB^4$

Suppose K bounds PL disks D in $\mathbb{Z}HB^4$ Z w/ one cone singularity

$Z \setminus$ nbhd(cone pt) : $S^3 \longrightarrow Y$

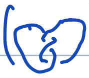
$D \setminus$ nbhd(cone pt) : $J \longrightarrow K$. $\overset{\text{knot}}{\text{Concordance}}$ inside $Z \setminus$ nbhd(cone pt).

$S^3_{\frac{1}{n}}(J) \overset{\text{Homological cob}}{\sim} Y_{\frac{1}{n}}(K)$ $\forall n$

d-invariants: • $d(Y) = d(Y')$ if Y_i are homologically cobordant.

• $d(S^3_{\frac{1}{n}}(Q)) = \begin{cases} -2V_0(Q) & n>0 \\ 0 & n=0 \\ 2V_0(\bar{Q}) & n<0 \end{cases}$

$d(S^3_{\frac{1}{2}}(J)) = d(S^3_{\frac{1}{3}}(J))$

$d(Y_{\frac{1}{n}}(K)) = d(S^3_{\frac{1}{n-2}}($  $))$

$d(Y_{\frac{1}{2}}(K)) = 0$

$d(Y_{\frac{1}{3}}(K)) = d(\text{Poincaré homological sphere}) = -2 \neq 0$ Contradiction //

L26 16th, Apr, 2025

Last time $\varepsilon: K \to (a_i)$ by $\varepsilon(K) = \begin{cases} +1 & a_1 > 0 \\ -1 & a_1 < 0 \\ 0 & \text{o/w} \end{cases}$

ε plays a role in understanding τ of satellites, in particular, cables.

Recall: $\Delta_{P(K)}(t) = \Delta_K(t^w) \cdot \Delta_{P(U)}(t)$, $w =$ |winding number|.

Meta Q: How do knot invariant behave under satelliting?

Recall: $g(K) = |w|\, g(K) + g(P \subset S^1 \times D^2)$, while for g_4 set. ε.

Q: How does τ behave under cabling?

Thm: $\tau(K_{p,q})$ depends on p, q, $\tau(K)$, $\varepsilon(K)$.

1) if $\varepsilon(K) = 1$, then $\tau(K_{p,q}) = p\,\tau(K) + (p-1)(q-1)/2$

2) if $\varepsilon(K) = -1$, then $\tau(K_{p,q}) = p\,\tau(K) + (p-1)(q+1)/2$

3) if $\varepsilon(K) = 0$, then $\tau(K) = 0$ and

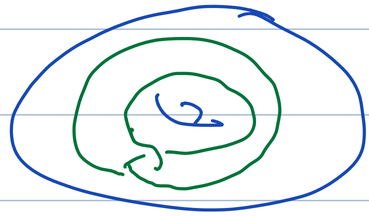
$$\tau(K_{p,q}) = \tau(T_{p,q}) = \begin{cases} \dfrac{(p-1)(q-1)}{2} & \text{if } q > 0 \\ \dfrac{(p-1)(q+1)}{2} & \text{if } q < 0 \end{cases}$$

Another application of ε.

Recall $P: \mathcal{C} \to \mathcal{C}$, $[K] \longmapsto [P(K)]$ well-defined: $K_0 \sim K_1 \Rightarrow P(K_0) \sim P(K_1)$

For which P is the map surjective? Injective? Bijective?

Ex $P =$



Whitehead Double.

$\Delta_{Wh(K)}(t) = 1 \Rightarrow$ not surjective:

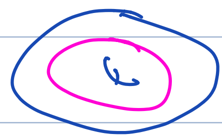
$K_0 \sim K_1 \Rightarrow \exists f, g.$ s.t. $\Delta_{K_0}(t) f(t) f(t^{-1}) = \Delta_{K_1}(t) g(t) g(t^{-1})$.

Another way to see not surjective: $g(Wh(K)) = 1$.

But being concordant to genus 1 knot $\Rightarrow$ slice genus at most 1 and $\exists$ knot w/ arbitrary large genus.

Exercise If $w(P) \neq \pm 1$, then P is not surjective.

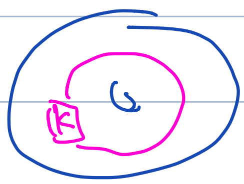
Ex. $P =$



unknot

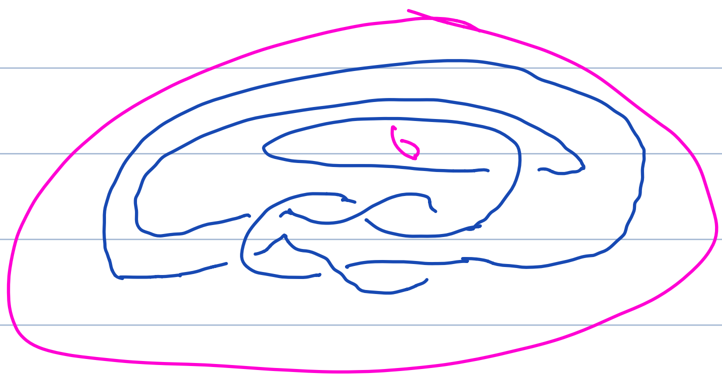
Induced map is id, surjective.

Ex

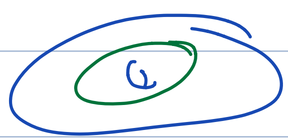


is also surjective.

Ex

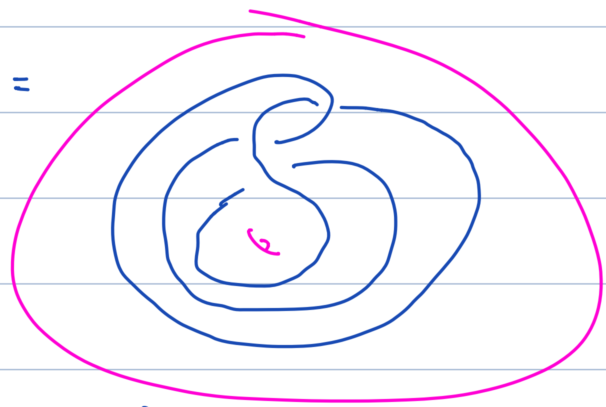


Concordant is $(S^1 \times D^2) \times I$ to



Q: $\exists$ non surjective winding number 1 pattern?

Ex

$Q =$ 

'Mazur pattern'

Thm: The Mazur pattern is not surjective.

Prop

$$\tau(Q(K)) = \begin{cases} \tau(K) & \text{if } \tau(K) \leq 0 \text{ and } \varepsilon(K) = 0 \text{ or } 1 \\ \tau(K)+1 & \text{if } \tau(K) > 0 \text{ and } \varepsilon(K) = -1 \end{cases}$$

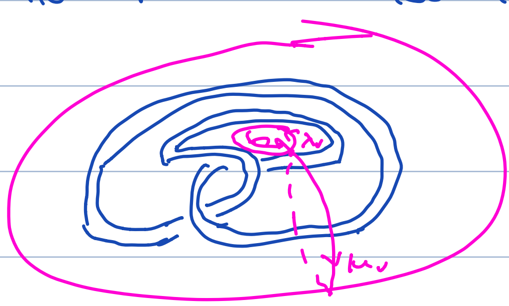
and

$$\varepsilon(Q(K)) = \begin{cases} 0 & \text{if } \tau(K) = \varepsilon(K) = 0 \\ 1 & \text{o/w} \end{cases}$$

So $\varepsilon(Q(K))$ never -1 $\Rightarrow$ Q not surjective.

Q: what about bijective P? (besides #) Yes.

Miller - Piccirillo 'Knot traces and concordance'



$\leftarrow P \subset S^1 \times D^2 =: V.$

λ_P = unique framing of P homologous to positive multiple of λ_V in $V - \nu(P)$ = Seifert framing of $P(U)$.

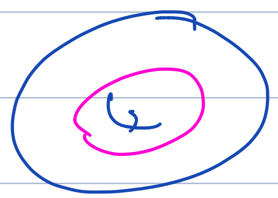
Def A pattern $P \subseteq V$ is dualizable if $\exists P^* \subset V^*$ s.t. $\exists$ orientation reversing homeo $h: V - \nu(P) \to V^* - \nu(P^*)$

w/ $h(\lambda_V) = \lambda_{P^*}$, $h(\mu_V) = -\mu_{P^*}$,

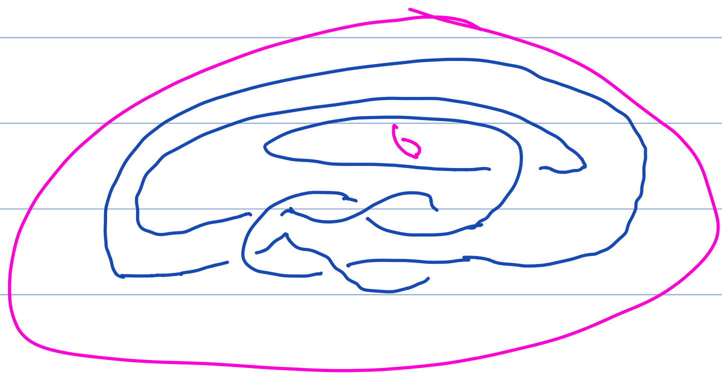
$h(\lambda_P) = \lambda_{V^*}$, $h(\mu_P) = -\mu_{V^*}$.

P^* is the dual of P.

Ex



is dualizable



also dualizable

Given any embedding $D^2 \to S^2$. $\exists$ embedding $S^1 \times D^2 \to S^1 \times S^2$

$\Rightarrow P \subseteq S^1 \times D^2$ induces $\hat{P} \subseteq S^1 \times S^2$

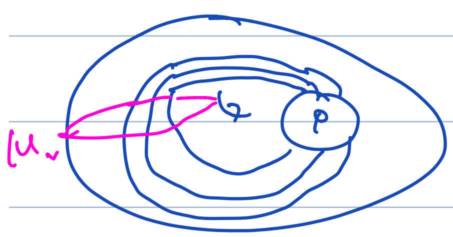
Equivalently, $S^1 \times D^2$ is one of handlebodies in a Heegaard splitting of $S^1 \times S^2$

Prop $P \subset V$ is dualizable $\iff \hat{P}$ is isotopic to $\hat{\lambda}_V$ in $S^1 \times D^2$.

Pf ($\Leftarrow$) Let $V^* = (S^1 \times S^2) - \nu(P^*)$

Exercise: $P^* = \hat{\lambda}_V \subset V^*$

($\Rightarrow$) $M := S^1 \times S^2 - \nu(\hat{P}) =$ Dehn filling of $V - \nu(P)$ along μ_V.

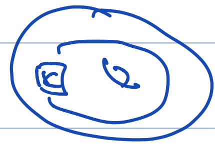


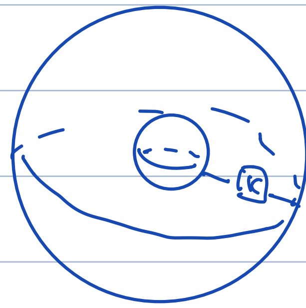
P dualizable $\Rightarrow$ M homeo to Dehn filling of $V^* - \nu(P^*)$ along μ_{P^*}.

This is just V^*.

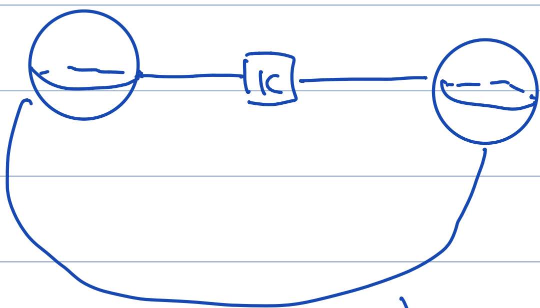
$\Rightarrow \hat{P}$ is a knot in $S^1 \times S^2$ w/ solid torus complement

By a result of Waldhausen, $\hat{P}$ is isotopic to $\pm \hat{\lambda}_V$ (i.e. S^1 fibre in $S^1 \times S^2$) //

Ex $K_\# =$ 

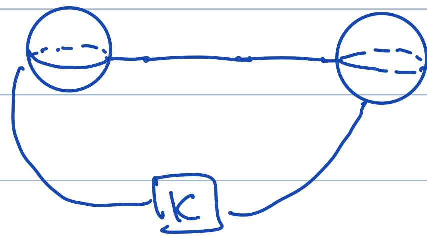


identify inner w/ outer spheres.

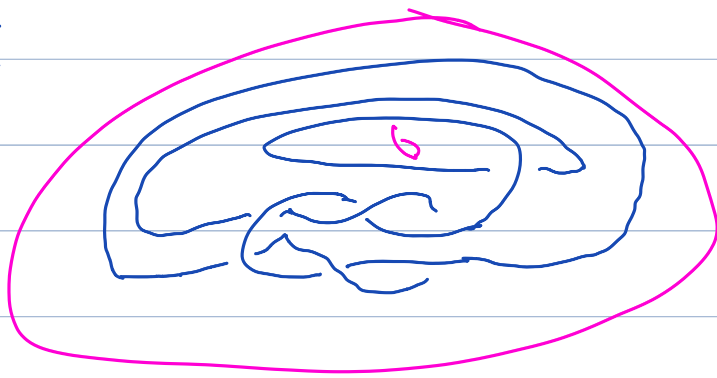


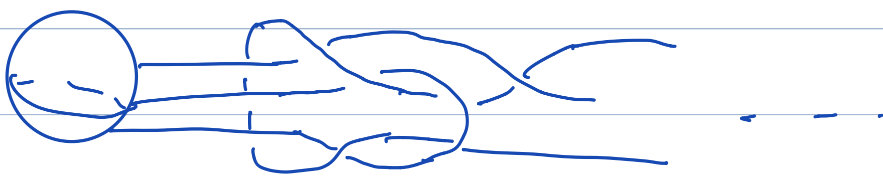
λ_V

Pulling sphere along K $\longrightarrow$



Ex





isotopy $\Rightarrow$