Math 8803

Monday pg 2.

Last time:

- dualizable patterns

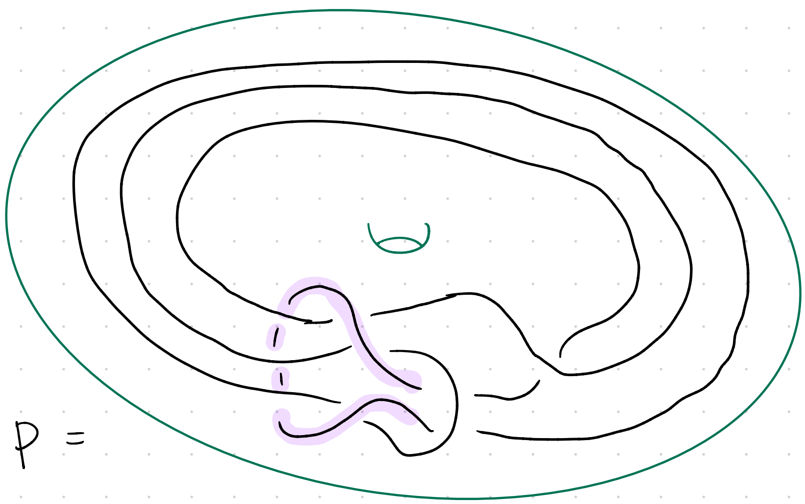
Goal: $P : \mathcal{C} \to \mathcal{C}$

s.t. $\exists\ Q : \mathcal{C} \to \mathcal{C}$ with $Q(P(K)) \sim K$

Reminder: April exercises due
Complete CIOS

Example:

$P =$



this pattern is dualizable.

Can see that by pulling strand along a meridinal disk to front and undo the knot.

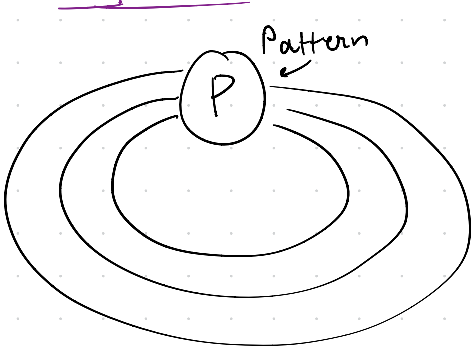
P^*

Theorem:

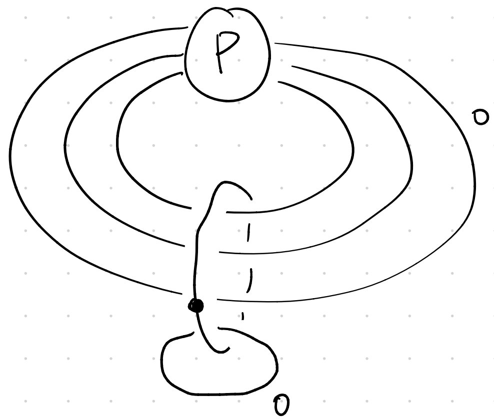
zero trace

$$\boxed{X_0(P(U)) \cong X_0(P^*(U))}$$

Proof idea:



Kirby diagram for a 4-mfd:



By exercise below, we get the theorem.

Exercise: 1-handle and either 2-handle give B^4

is B^4

Remaining 2-handle is attached along $P(U)$ or $P^*(U)$.

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Proposition

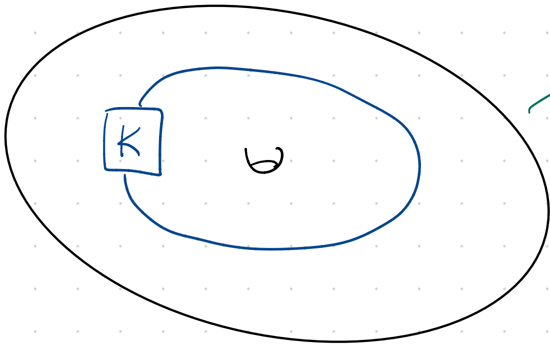
If P and Q are dualizable, then $P \circ Q$ is dualizable with dual $Q^* \circ P^*$

Proof: left as an exercise

Let $\bar{P}$ be the pattern obtained from $P \subset S^1 \times D^2 = V$ by reversing orientation of V and string orientation

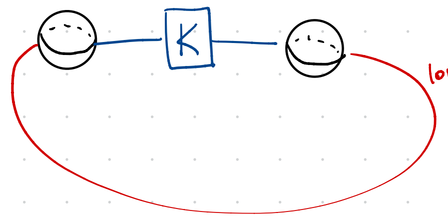
(ie. change all crossings) in diagram

(ie. change direction of arrow) in diagram

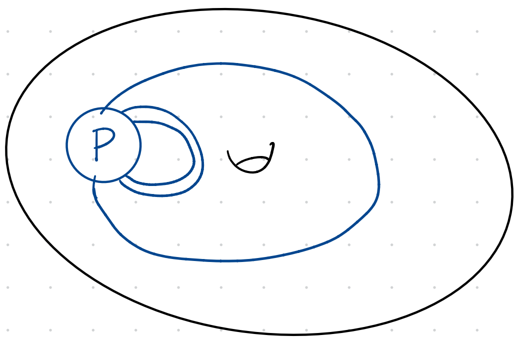
$K_\# =$ 

dual of $K_\#$ is $K_\#$

connect sum



$P_\# = P(U)_\#$

$P_\# =$ 

Corollary

For P dualizable, $\overline{P}^*(P(U)) \sim U \sim P(\overline{P}^*(U))$

Proof:

$$X_0(\overline{P}^*(P(U))) = X_0(\overline{P}^* \circ P_\#(U))$$

$$= X_0((\overline{P}^* \circ P_\#)^*(U))$$

bar and star commute and $P_\#^* = P_\#$

$$= X_0(P_\# \circ \overline{P}(U))$$

$$= X_0(P(U) \# \overline{P}(U))$$

$P(U) \# \overline{P}(U)$ is slice.

By trace embedding lemma, this embeds in S^4

$$X_0(P(U) \# \bar{P}(U)) \text{ embeds in } S^4$$

$$\Rightarrow X_0(\overline{P^*}(P(U))) \text{ embeds in } S^4$$

$$\overset{\text{T.E.L}}{\Rightarrow} \overline{P^*}(P(U)) \text{ is also slice.}$$

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Theorem (Miller-Piccirillo, reproof of Gompf-Miyazaki)

$$\overline{P^*}(P(K)) \sim K \sim P(\overline{P^*}(K))$$

Proof:

obvious candidate when we want something slice like last thm

$$\overline{P^*}(P(K)) \# -K = (\overline{K_\#} \circ \overline{P^*})(P \circ K_\#)(U)$$

$$= (\overleftarrow{K_\# \circ P^*})(P \circ K_\#)(U)$$

$K_\# \circ P^*$ is the dual of $P \circ K_\#$, so by previous corollary,

$$(\overleftarrow{K_\# \circ P^*})(P \circ K_\#)(U) \text{ is slice.}$$

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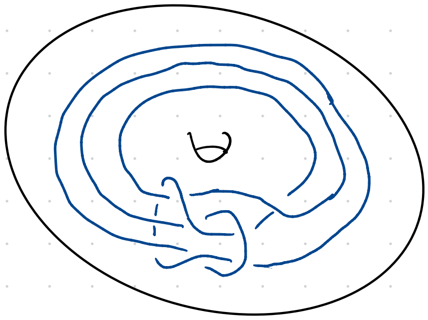
Theorem (Miller-Piccirillo)

$\exists$ infinitely many pairs of knots (K_n, K_n') with diffeomorphic 0-trace but K_n, K_n' are not concordant, even up to orientation reversal

Proof: $P =$ 

$T_n(P) = P$ with n full twists added

$K_n = T_{2n-1}(P)$

$K_n' = T_{-2n-3}(P)$

Exercise:

$$\boxed{\text{general fact: For any dualizable pattern } P,\ T_n(P) \text{ is dualizable with dual } T_{-n}(P^*)}$$

Exercise: $X_0(K_n) = X_0(K_n')$

Now we want to show that $K_n \not\sim K_n'$:

relies on double branched covers

Let $\Sigma_2(K) =$ double branched cover of S^3 branched over K

Proposition

If K is slice, then $\Sigma_2(K)$ bounds a $\mathbb{Q}HB^4$

Proof:

Exercise: If D is a slice disk for K, then a double branched cover of B^4 branched over D is a $\mathbb{Q}HB^4$ with boundary $\Sigma_2(K)$

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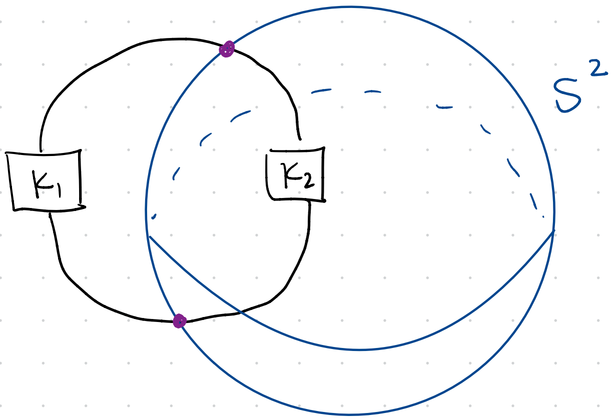
Proposition: $\Sigma_2(K_1 \# K_2) = \Sigma_2(K_1) \# \Sigma_2(K_2)$

Remark: $S^3_{+1}(K_1 \# K_2) \neq S^3(K_1) \# S^3_1(K_2)$ in general

Proof:



Consider the connect sum S^2 that intersects $K_1 \# K_2$ in exactly 2 points. The double branched cover of S^2 branched over 2 points is S^2.

Together these two propositions tell us the following:

we obtain a homomorphism
$$\mathcal{C} \longrightarrow \Theta^3_{\mathbb{Q}}$$
$$[K] \longmapsto [\Sigma_2(K)]$$

where $\Theta^3_{\mathbb{Q}}$ is the rational homology cobordism group

$\mathbb{Q}HS^3$'s Y_0, Y_1 are $\mathbb{Q}$-homology cobordant if $\exists$ smooth compact W^4

1. $\partial W = Y_0 \sqcup Y_1$
2. $\iota_*: H_*(Y_i; \mathbb{Q}) \xrightarrow[\text{isom}]{} H_*(W; \mathbb{Q})$

Exercise: $\left| H_1(\Sigma_2(K); \mathbb{Z}) \right| = \left| \Delta_K(-1) \right|$

primitive 2^{nd} root of unity ...

why -1?
looking @ Σ_2 and

Upshot: can use invariants of $\mathbb{Q}$-homology cobordisms to obstruct concordance.

d-invariants are invariants of $\mathbb{Q}$-homology cobordism:

If W is a $\mathbb{Q}$-homology cobordism between $\mathbb{Z}HS^3$'s Y_0 and Y_1, then $d(Y_0) = d(Y_1)$

Proposition

$\Sigma_2(K_n)$ and $\Sigma_2(K_n')$ are $\mathbb{Z}HS^3$'s and

$$d(\Sigma_2(K_n')) = -2$$
$$d(\Sigma_2(K_n)) = 0$$

Exercise: Use the proposition to show that $P: \mathcal{C} \longrightarrow \mathcal{C}$ is not given by connected sum.

BIG IMPORTANT OPEN PROBLEMS IN HOMOLOGY COBORDISM

1. Slice ribbon conjecture

   Q: Is every slice knot ribbon?

2. Smooth 4-D Poincaré conjecture

   Q: Is every smooth 4-manifold (closed, simply connected) homotopy equiv to S^4 diffeomorphic to S^4?

3. Does there exist a non-slice knot $K \subset S^3$ such that K bounds a smooth slice disks in

   a) homotopy B^4?

   b) $\mathbb{Z}HB^4$?

<u>Note</u>: a) would give a disproof of smooth 4-D poincaré conj.

<u>Remark</u>: $\exists$ non-slice knots in S^3 that do bound smooth slice disks in a $\mathbb{Q}HB^4$

<u>Example</u>: • 4_1 not slice but bounds disk in a $\mathbb{Q}HB^4$

- more generally, any <u>strongly negative amphichiral knot</u>

  $\exists$ orientation reversing homeo $\phi: S^3 \longrightarrow S^3$ s.t. $\phi(K) = K$ and ϕ has exactly 2 fixed pts on K

4. Consider $\ker\left(\Theta^3_{\mathbb{Z}} \longrightarrow \Theta^3_{\mathbb{Q}}\right)$

i.e. $\mathbb{Z}HS^3$'s that bound $\mathbb{Q}HB^4$'s

Ex: $\Sigma(2,3,7) \in \ker\left(\Theta^3_{\mathbb{Z}} \to \Theta^3_{\mathbb{Q}}\right)$

Q: Is this kernel infinitely generated?

5. Ribbon concordance from K_0 to K_1 is a concordance with no local maxima.

Conjecture: (Gordon) Ribbon concordance is a partial order.

Resolved in the affirmative by Agol

Q: Given a knot K, what can we say about the partially ordered set $[K]$?

Exercise: If $K_0 \sim K_1$, $\exists$ K_2 such that $\exists$ a ribbon concordance K_0 to K_2 and ribbon concordance K_1 to K_2

$K_0 < K_2$ $\qquad$ $K_1 < K_2$

- Is the order type of $[K]$ independent of K?
- $\exists$ infinite descending chains $K_0 > K_1 > K_2 > K_3 > \dots$?
- Does every concordance class contain a unique minimal element?

6. $\exists$ torsion in $\Theta^3_{\mathbb{Z}}$?

7. Is all torsion in $\Theta^3_{\mathbb{Z}}$ generated by negatively amphichiral knots?