8803

Week 2 Notes

monday Jan 13 pg 2
wednesday Jan 15 pg 11

January 13 2025

Last time:

- Seifert form
- S-equivalence class is a knot invariant

(★) K slice ⇒ Seifert form is metabolic $\left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$

vanishes on half dim. subspace

Alexander Polynomial depends on S-equivalence class

$$\Delta_K(t) = \det(V - tV^T)$$

Exercise: $\Delta_K(1) = \pm 1$

Corollary: Fox–Milnor Condition

If K is slice, then $\Delta_K(t) = p(t)p(t^{-1})$ for some $p(t) \in \mathbb{Z}[t]$

proof:

Since K is slice, (★) implies $\exists$ Seifert surface matrix for K of the form $V = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$.

Hence $\det(V - tV^T) = \det \left[\begin{array}{c|c} A - tA^T & B - tC^T \\ \hline C - tB^T & 0 \end{array}\right]$ all square matrices

$$= -\det(B - tC^T)\det(C - tB^T)$$

(B is a g×g matrix)

$$= (-t)^g \underbrace{\det(B - tC^T)}_{p(t)} \underbrace{\det(B - t^{-1}C^T)}_{p(t^{-1})}$$

///

Note: A polynomial is a more tractable invariant than the S-equivalence class

Defn: the **determinant** of a knot K $\left|\Delta_K(-1)\right|$

(implies that determinant is always odd)

Corollary:

$\boxed{\text{If } K \text{ is slice, then determinant, } \det K\text{, is a perfect square}}$

Proof: $\Delta_K(-1) = p(-1)\,p(-1^{-1}) = \underline{p(-1)\,p(-1)}$ ///

Recall our goal was to obstruct sliceness.

Example:

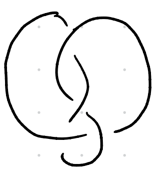
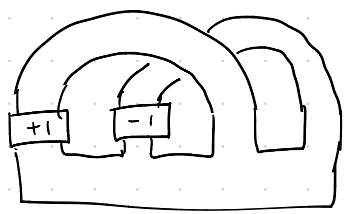


Figure 8 =



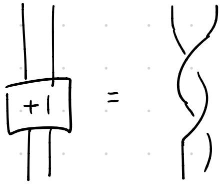
$$\Delta_K(t) = t - 3 - t^{-1}$$

$\left|\Delta_K(-1)\right| = 5 \implies 4_1$ is not slice

$\implies 4_1$ is order 2 in $\mathcal{C}$

Note: converse of corollary is not true and in general invariants do not detect sliceness.

Remark:



Corollary:

$\boxed{\text{If } K \text{ is slice, then } \sigma(K) = 0}$

proof:

$$(\star) \Rightarrow V = \left[\begin{array}{c|c} A & B \\ \hline C & 0 \end{array}\right]$$

$$V + V^T = \begin{pmatrix} A + A^T & B + C^T \\ C + B^T & 0 \end{pmatrix}$$

Exercise:

1. signature of $\uparrow$ is zero

2. signature of a knot, $\sigma(K)$ is always zero

Corollary

$\boxed{\sigma : \mathcal{C} \longrightarrow 2\mathbb{Z} \text{ is a surjective homomorphism}}$

proof:

- homom. since σ is additive, i.e. $\sigma(K_1 \# K_2) = \sigma(K_1) + \sigma(K_2)$
- signature of a slice knot is zero

- $\sigma(\underbrace{nRHT}) = -2n$ (since concordance group $\mathcal{C}$ is Abelian)

$nRHT = \underbrace{RHT \# \cdots \# RHT}_{n \text{ times}}$

Q: $\sigma(4_1) = 0$ ... $\sigma(\underbrace{4_1 \# 4_1}_{\text{slice}}) = 0$

Note: 4_1 = figure 8

So $2\sigma(4_1) = 0$

Corollary

$\mathcal{C}$ contains a $\mathbb{Z}$-summand, i.e. $\mathcal{C} \cong \mathbb{Z} \oplus G$ for some Abelian group G.

proof:

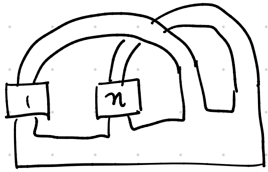
(Note: $\twoheadrightarrow$ surjective)

$$0 \longrightarrow \ker\sigma \longrightarrow \mathcal{C} \overset{\sigma/2}{\twoheadrightarrow} \mathbb{Z} \longrightarrow 0$$

short exact sequence

Since $\mathbb{Z}$ is free, you can define a section s so s.e.s. splits as a direct sum, $\mathcal{C} \cong s(\mathbb{Z}) \oplus \ker\sigma$

Note: RHT is infinite order in $\mathcal{C}$

Q: Are these knots  linearly independent in $\mathcal{C}$?

Levine–Tristram Signatures: $\sigma_\omega(K)$

Recall that a Hermitian matrix $\left(A = \overline{A^T}\right)$ is diagonalizable.

V Seifert matrix

$$V_\omega := (1-\omega)V + (1-\overline{\omega})V^T \qquad \omega \in \mathbb{C} \quad |\omega| = 1$$

Fact: If ω is not a root of $\Delta_K(t)$, then this form is nonsingular.

Def'n: $\sigma_\omega(K) := \operatorname{sgn} V_\omega$

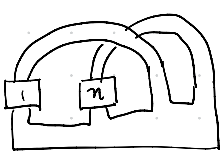
(More precisely, can define $\sigma_\omega(K)$ to be the average of limits σ_{ω^+} and σ_{ω^-})

Like the ordinary signature, it's additive under connected sum and vanishes on slice knots.

$$\sigma_\omega(K_1 \# K_2) = \sigma_\omega(K_1) + \sigma_\omega(K_2)$$

$$\sigma_\omega(\text{slice knot}) = 0$$

Exercise:
Use σ_ω to give a surjective map $\mathcal{C} \longrightarrow \mathbb{Z}^\infty$

Hint: consider  for certain n

Corollary:

$\boxed{\mathcal{C} \text{ is infinitely generated}}$

proof: surjection $\mathcal{C} \longrightarrow \mathbb{Z}^\infty$

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Note: if K is finite order in $\mathcal{C}$, then $\sigma_\omega(K) = 0$

Arf Invariant

- Can define in terms of the Seifert form
- Define a $\mathbb{Z}/2$ - valued quadratic form on $(\mathbb{Z}/2)^{2g}$ by

$$q(x) = xVx^T$$

$$\mathrm{Arf}(q) = \begin{cases} 0 & \text{if } q \text{ takes on value } 0 \text{ on majority of elmts in } (\mathbb{Z}/2)^{2g} \\ 1 & \text{if } q \text{ " } \quad \text{" } 1 \text{ " } \quad \text{" in } (\mathbb{Z}/2)^{2g} \end{cases}$$

Exercise:

$\mathrm{Arf}: \mathcal{C} \longrightarrow \mathbb{Z}/2$

[More details found in Livingston survey]

Fact: $\mathrm{Arf}\ K = 0 \iff \Delta_K(-1) = \pm 1 \mod 8$

Remark: $K \rightsquigarrow V \rightsquigarrow q \xrightarrow{\mathrm{Arf}} \mathbb{Z}/2$

More $\mathbb{Z}/2$ - valued concordance invariants

Def'n: A symmetric polynomial is a polynomial $p(t)$ s.t.

$$p(t^{-1}) = \pm t^n p(t)$$

Example: $p(t) = t^2 - t + 1$ is symmetric since

$$p(t^{-1}) = t^{-2} - t^{-1} + 1 \quad \text{and} \quad t^2 p(t^{-1}) = p(t).$$

Recall: Fox-Milnor condition: If K is slice, then $\Delta_K(t) = p(t)\,p(t^{-1})$ for some $p(t) \in \mathbb{Z}[t]$

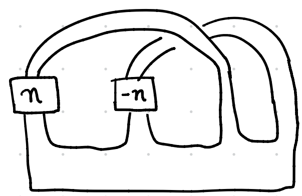
If $p(t)$ is an irreducible (over $\mathbb{Z}$) symmetric polynomials then exponent of $p(t)$ mod 2 in an irreducible factorization of $\Delta_K(t)$ is a surjective homomorphism

$$\mathcal{C} \longrightarrow \mathbb{Z}/2$$

Example: $\Delta_{4_1}(t) = t^2 - 3t + 1$

Exercise: Show that $\mathcal{C}$ has a $(\mathbb{Z}/2)^\infty$ direct summand

Hint: consider

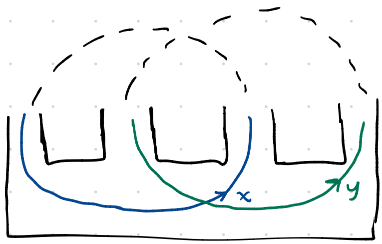


← genus 1 knot

(these knots are negative amphichiral)
(compute $\Delta_K(t)$ for this family and their irreducible factorization)

Q: Is there a unified approach to extracting concordance invariant from the Seifert form?

A: Yes but we need a few things:



intersection form on $H_1(F)$

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Exercise: $V - V^T =$ intersection form on $H_1(F)$

Defn: an abstract Seifert form is a bilinear form (that looks like it could be the Seifert form of a knot)

$$V : \mathbb{Z}^{2n} \times \mathbb{Z}^{2n} \longrightarrow \mathbb{Z} \qquad \text{s.t. } V - V^T \text{ is unimodular}$$

Fact: Every abstract Seifert form can be realized by some Seifert surface for some knot

Recall: $K \rightsquigarrow V$ Seifert form (up to s-equivalence)

$-K \rightsquigarrow -V$

$K_1 \# K_2 \rightsquigarrow V_1 \oplus V_2$

K slice $\Longrightarrow$ V is metabolic

$K_0 \sim K_1 \Leftrightarrow K_0 \# -K_1$ slice, define $V_0 \sim V_1 \Leftrightarrow V_0 \oplus -V_1$ metabolic

Exercise: check above is an equivalence relation $V_0 \sim V_1$

$$\boxed{\mathcal{G}} := \left(\{\text{abstract Seifert forms}\}/\sim \;,\; \underset{\text{direct sum}}{\oplus} \right)$$

$\shortparallel$

Algebraic Concordance group

$$\mathcal{C} \twoheadrightarrow \mathcal{G} \quad \text{is a well-defined surjective homomorphism}$$
$$[K] \longmapsto [V]$$

by above exercise on abstract Seifert forms

Advantage of this approach: studying $\mathcal{G}$ utilizes linear algebra

[Levine 1969]: $\mathcal{G} \cong \underbrace{\mathbb{Z}^\infty \oplus \mathbb{Z}/_2{}^\infty}_{\text{we've discussed this}} \oplus \underbrace{\mathbb{Z}/_4{}^\infty}_{\text{bit more complicated.}}$

Q: Does $\mathcal{C} \twoheadrightarrow \mathcal{G}$ have kernel?

[Casson-Gordon 1975] $\ker \mathcal{C} \to \mathcal{G}$ nontrivial

Remark: Can study concordance in higher dimensions

i.e. knotted S^n in S^{n+2}

In higher odd dimensions, $\mathcal{C} \cong \mathcal{G}$

A big open question:

Is there n-torsion in $\mathcal{C}$ for $n \neq 2$?

i.e. only 2 torsion is known and basically realized by negative amphichiral knot

January 15, 2025

Last time:

- algebraic concordance group

$$\mathcal{C} \twoheadrightarrow \mathcal{G} \cong \mathbb{Z}^\infty \oplus \mathbb{Z}/_2{}^\infty \oplus \mathbb{Z}/_4{}^\infty$$

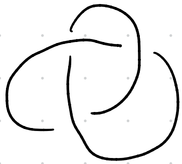
$$\mathcal{C} \overset{?}{=} \mathbb{Z}^\infty \oplus \mathbb{Z}/_2{}^\infty \oplus G$$

Today:

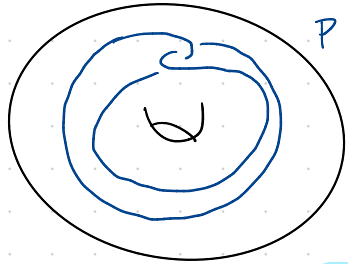
Satellite Knots

How to build new knots out of ones we already know?

we already know about connect sum.



K



P

$S^1 \times D^2$

this particular pattern is called a **Whitehead double** $Wh(K)$

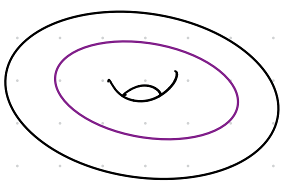
$$h : S^1 \times D^2 \longrightarrow \nu(K)$$

longitude $S^1 \times \{x\} \longmapsto$ 0-framed longitude of K
$x \in \partial D^2$

↳ links K zero times

↳ boundary of a Seifert surface for K

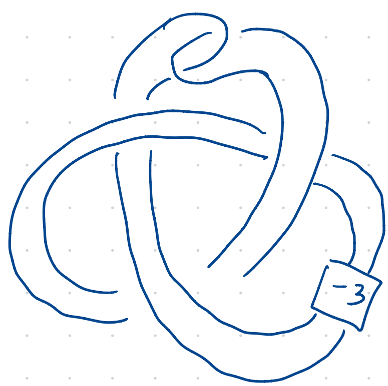
x



So for this example,

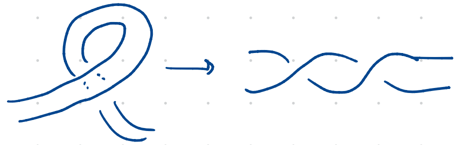
K

writhe $wr(D) = +3$

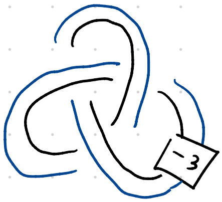


$$P(K) := h(P)$$

Where does the -3 come from?



Now the linking number between blue and black is zero:



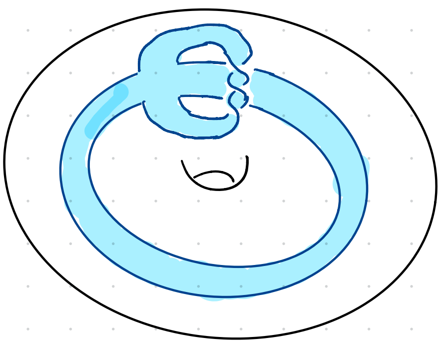
"blackboard framing"

Exercise:

the Seifert form of a Whitehead double $Wh(K)$ is

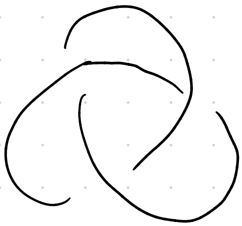
$$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

Correction: figure has an extra half twist that shouldn't be there



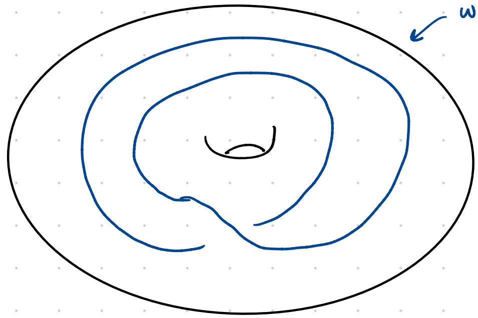
Corollary

$$\boxed{\Delta_{Wh(K)}(t) = 1}$$



Companion

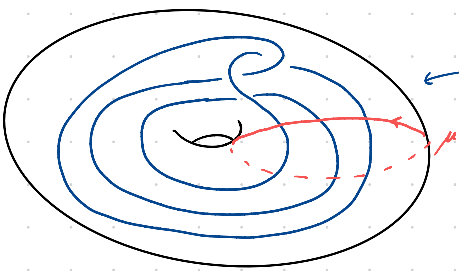
$w(P) = 2$



$(2,1)$-cable

(p,q)-cable denoted $K_{p,q}$

$w(K_{p,q}) = p$



$w(P) = 1$

Mazur Pattern

μ is the meridian

The winding number of P $\quad w(P) := lk(P, \mu)$

Equivalently $\quad w(P) = [P] \in H_1(S^1 \times D^2; \mathbb{Z})$

Q: How can we build a Seifert surface from a satellite knot $P(K)$?

[P = "pattern" and P could be whitehead double

If $w(P) = 0$, then P bounds a Seifert surface F in solid torus $S^1 \times D^2$

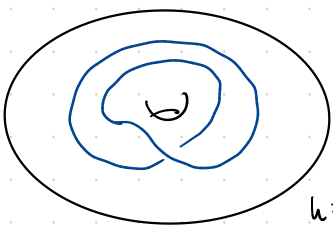
and then $h(F)$ is a Seifert surface for $P(K)$

$$h: S^1 \times D^2 \longrightarrow \nu(K)$$

If $w(P) \neq 0$, then P is not null-homologous in $S^1 \times D^2$ and hence does not bound a surface in $S^1 \times D^2$

However, $P \amalg w(P)$-longitudes does bound a surface in $S^1 \times D^2$

Example:



Surface S

$$h: S^1 \times D^2 \longrightarrow \nu(K)$$

$$S^1 \times D^2 \cup_{h|_\partial} S^3 - \nu(K)$$

Exercise:

Modify Seifert's algorithm to construct such a surface in general

Build a Seifert surface $P(K)$ via $S \cup w(P)$ parallel copies of F, where F is a Seifert surface for K

Theorem (Schubert 1953)

If S and F are minimal genus such surfaces, then the resulting surface is a minimal genus Seifert surface for $P(K)$ (K nontrivial)

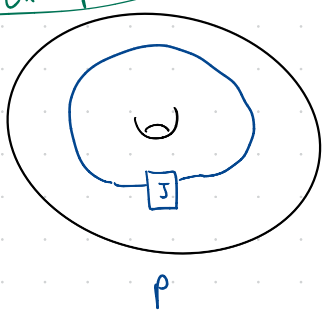
Exercise:

$w = w(P)$

$$\Delta_{P(K)}(t) = \Delta_K(t^w) \cdot \Delta_{P(u)}(t)$$

u = unknot

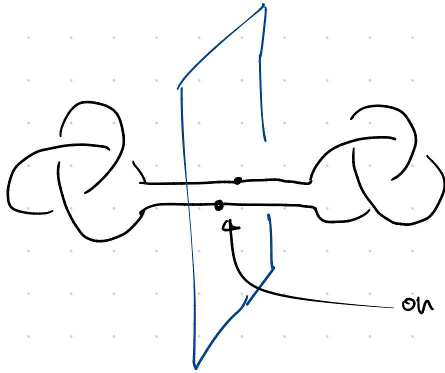
Example:



$P(K) = K \# J$

Recall: The key feature of a knot K that is a nontrivial connected sum (called a composite knot) is that $\exists$ a 2-sphere that intersects K in exactly two points such that on either side, the arc is knotted

↳ not isotopic to an arc in that 2-sphere

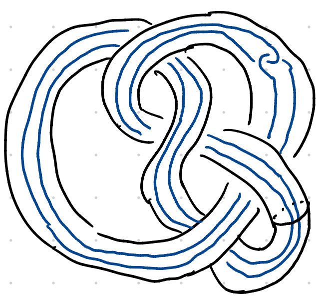


S^2

on either side we have knotted arcs

Key feature of a Satellite knot, J, is that $\exists$ an essential torus (non-boundary parallel, incompressable) in $S^3 - \nu(K)$ that is the boundary of the solid torus containing P

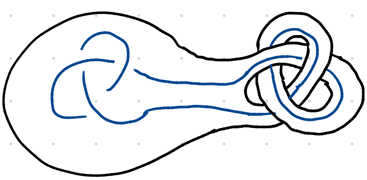
Example:



$J = P(K)$

essential torus

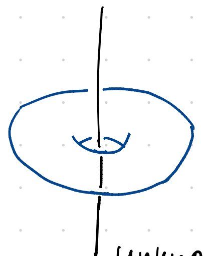
Example:



Swallow-follow

proof of Schubert's theorem involves studying the intersection of an essential with Seifert surface.

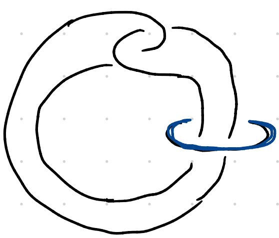
Observe: the complement of an unknot in S^3 is a solid torus.

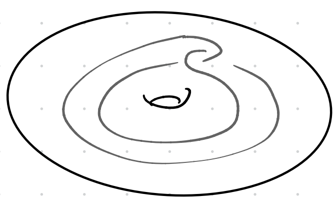


unknot

Consequence: any two component link with a distinguished unknotted component describes a pattern P in $S^1 \times D^2$

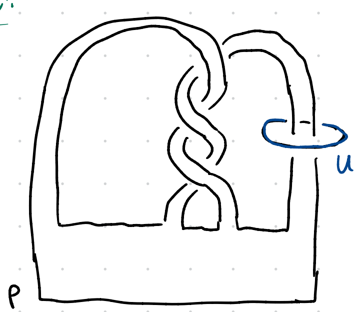
Example:





fact: can choose either component in this example

Example:



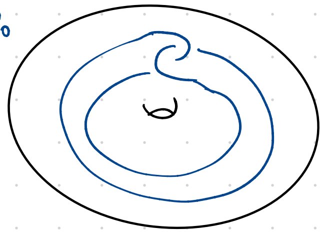
p lives in $S^1 \times D^2$ with a preferred longitude.

↓

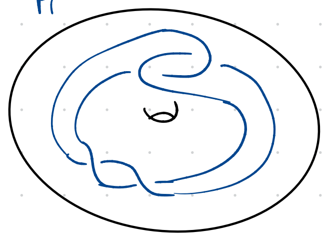
is the meridian of μ

Note:

P_0



P_1



by a Dehn Twist

$\exists$ a homeomorphism $h: S^1 \times D^2 \longrightarrow S^1 \times D^2$ s.t. $h(P_0) = P_1$

In this sense, they are equivalent knots in $S^1 \times D^2$
(under this Dehn Twist, the preferred longitude changed)

Exercise:
P_0 and P_1 are not ambiently isotopic in $S^1 \times D^2$

Upshot: Preferred longitude (or equivalently, identification of solid torus with $S^1 \times D^2$) matters