8803

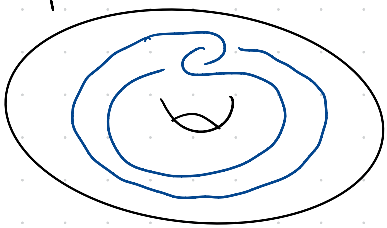
Week 3 Notes

Wednesday

Last time:

Satellite knots

pattern $P \subset S^1 \times D^2$



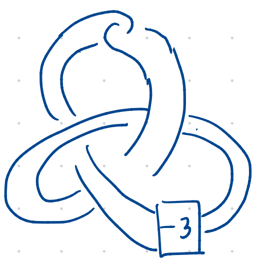
companion K

$P(K)$

$$h : S^1 \times D^2 \longrightarrow \nu(K)$$

$$S^1 \times \{x\} \longmapsto \text{0-framed longitude}$$

$x \in \partial D^2$



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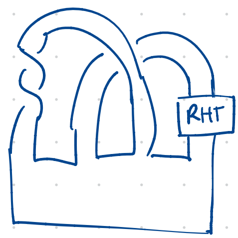
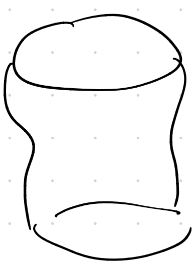


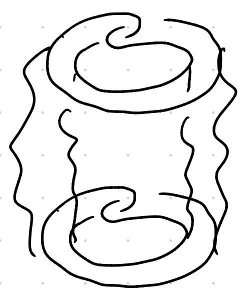
Q: What about satellite and concordance?

Observe: $K_0 \sim K_1 \implies P(K_0) \sim P(K_1)$



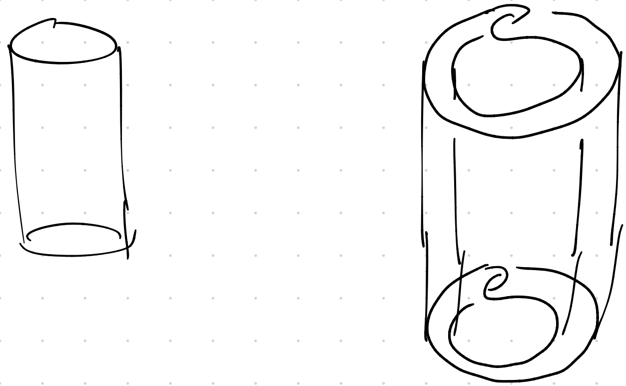
K_1

K_0



$P(K_1)$

"satellite the concordance"

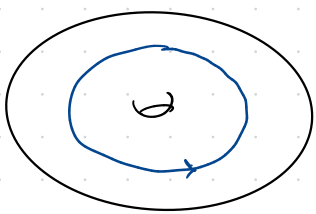


Hence: we get a well-defined map of sets

$$P: \mathcal{C} \longrightarrow \mathcal{C}$$
$$[K] \longmapsto [P(K)]$$

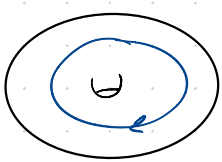
Q: Is this map a homomorphism?

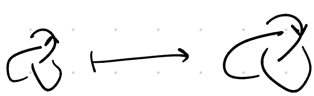
Example:

$P_0 =$ 

$P: \mathcal{C} \longrightarrow \mathcal{C}$ identity

Remark: Everything is oriented, but we could have the strand orientation change.

$P_0' =$ 

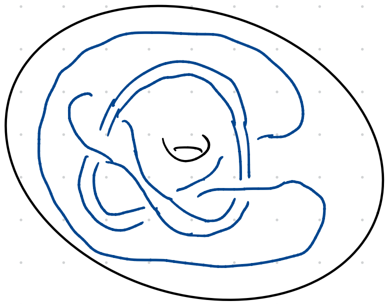


$$P: \mathcal{C} \longrightarrow \mathcal{C}$$
$$[K] \mapsto [K^r]$$

Fact: $\exists$ knots that are not concordant to their reverses

(Livingston 1981, using refinements of Casson-Gordan techniques)

Example:

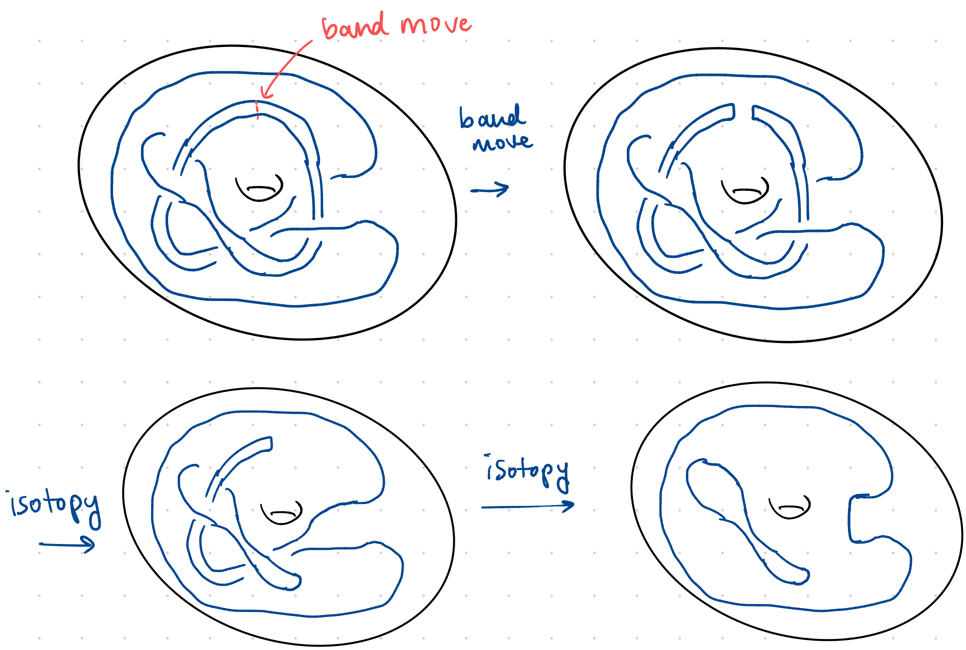
$P_1 =$ 

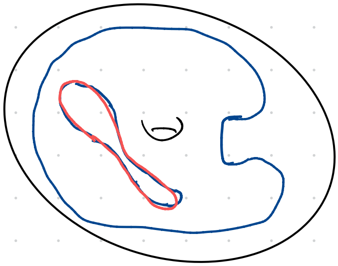
$P: \mathcal{C} \longrightarrow \mathcal{C}$

ribbon knot but the ribbon disk doesn't live in the solid torus

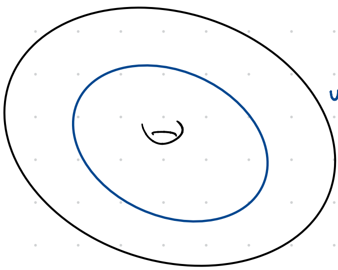
Note: this is a knot in $S^1 \times D^2$

Can embed solid torus in S^3 in standard way, so it looks ribbon-y, but just thinking of it in





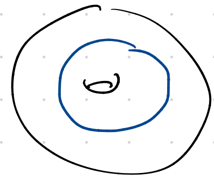
this component can cap off with a disk



which is P_0!

In $S^1 \times D^2 \times I$,

P_1 is concordant to



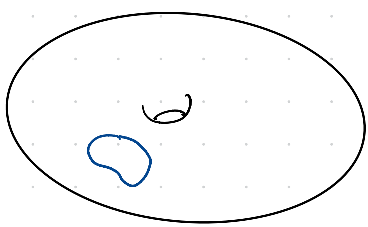
$$\Longrightarrow P_0(K) \sim P_1(K)$$

because in $S^1 \times D^2 \times I$, $P_1 \sim P_0$

When satelliting knots, it's important that the concordance stayed in $S^1 \times D^2 \times I$.

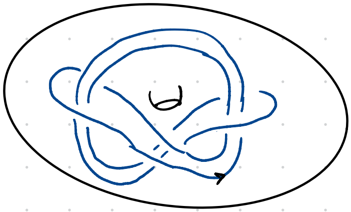
Other Patterns:

Example:

$P_2 =$ 

$P_2 : \mathcal{C} \longrightarrow \mathcal{C}$ 0-map

since $P_2(K)$ is the unknot $\forall K$

$P_3 =$ 

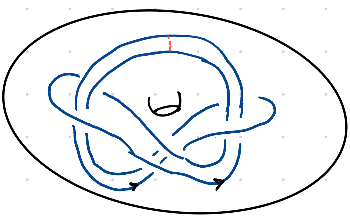
$P_3 : \mathcal{C} \longrightarrow \mathcal{C}$

Let's check the winding no. of this knot : $\omega(P_3) = 0$

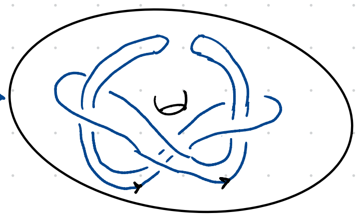
P_3 induces a map on set of knots in S^3 up to isotopy

$$P_3 = \{\text{knots in } S^3\} \longrightarrow \{\text{knots in } S^3\}$$

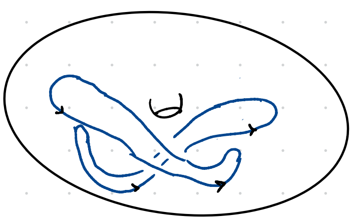
Let's do a band move:



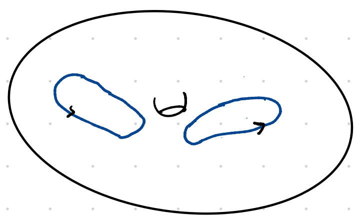
$\xrightarrow{\text{band move}}$



$\xrightarrow{\text{isotopy}}$



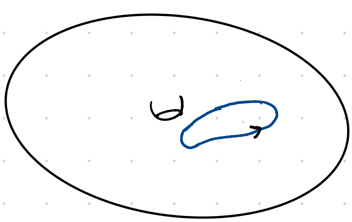
$\xrightarrow{\text{isotopy}}$



$P_3 : \mathcal{C} \longrightarrow \mathcal{C}$ 0-map

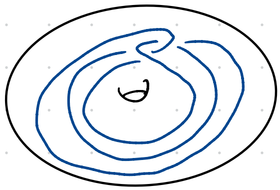
P_3 concordant in $S^1 \times D^2 \times I$ to P_2

$\xrightarrow{\text{cap off with a disk}}$



Rmk: Ribbon in the solid torus then it will induce the 0-map

Example:



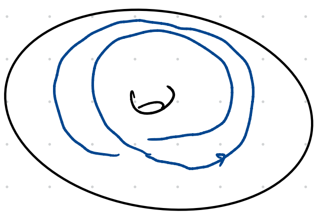
Mazur knot does not induce identity map on $\mathcal{C}$

(can show using Ozsváth-Szabó τ-invariant)

Q: Is the map determined by the winding no.? — No

Note: In general, $P: \mathcal{C} \longrightarrow \mathcal{C}$ is not a homomorphism

Example:

$P =$ 

(2,1)-cable

$P(K) = K_{2,1}$

$\exists$ a smooth concordance homomorphism coming from Heegaard Floer homology

$$\tau: \mathcal{C} \longrightarrow \mathbb{Z}$$

and $\tau(RHT_{2,1}) = 2$

$\tau(LHT_{2,1}) = -1$

(but LHT is a concordance inverse of RHT)

Hence $P_{2,1}: \mathcal{C} \longrightarrow \mathcal{C}$ is not a homomorphism

those two should have been inverses, but 2 and -1 are not

Conjecture: (Heddon, see also Pinzon-Caicedo, A. Miller)

$\boxed{\text{There are no interesting homomorphisms of } \mathcal{C} \text{ induced by satellite operations.}}$

the only possible homomorphisms are id, the reversal id^r, and the 0-map

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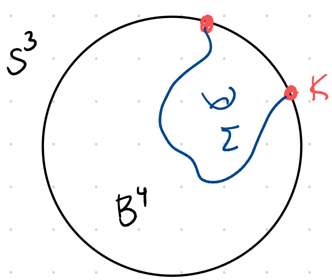
Slice genus, or 4-ball genus

def'n: the smooth/top slice genus (or 4-ball genus) of a knot $K \subset S^3$ is

$$g_4(K) = \min\{ g(\Sigma) \mid \Sigma \text{ a smooth/top locally flat surface in } B^4 \text{ with } \partial\Sigma = K \}$$



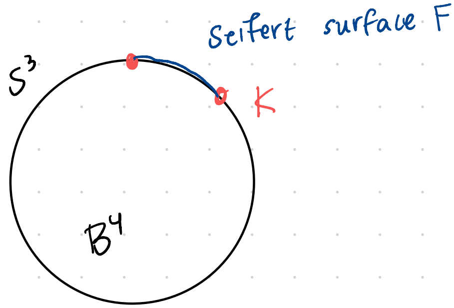
Notation:

use g_4 for smooth slice genus

use g_4^{TOP} for top slice genus.

Remark: K slice $\iff g_4(K) = 0$

Observe: $g_4(K) \leq g(K)$



can push F into B^4

but might not be the best one for minimal genus.

Example:

$$g_4(RHT) = g_4(LHT) = 1$$

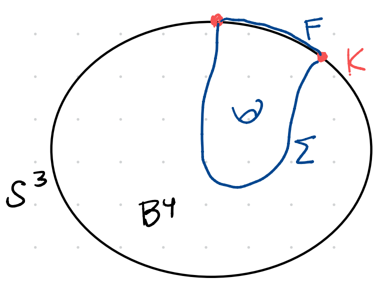
since RHT and LHT are not slice (so $g_4 \neq 0$)

and ordinary $g(RHT) = g(LHT) = 1$.

Exercise:

Prove $\left|\frac{\sigma(K)}{2}\right| \leq g_4^{TOP}(K)$

Hint:



closed surface $F \cup \Sigma$

Not going to get the same "half lives/half dies" but do get something close.

Example: $g_4(RHT \# RHT) = 2$

using σ

$\sigma(RHT \# RHT) = -4 \implies g_4^{TOP}(RHT \# RHT) \leq 2$

$g(RHT \# RHT) = 2$

Example: $g_4(\underline{nRHT}) = n$

$= \underbrace{RHT \# \cdots \# RHT}_{n \text{ times}}$

In general, slice genus is not additive under connected sum.

$$g_4(K_1 \# K_2) \neq g_4(K_1) + g_4(K_2)$$

example: $g_4(RHT) = g_4(LHT) = 1$

$g_4(RHT \# LHT) = 0$

Note: We can build a slice surface for a satellite knot in a similar fashion to how we built a Seifert surface for a satellite knot.

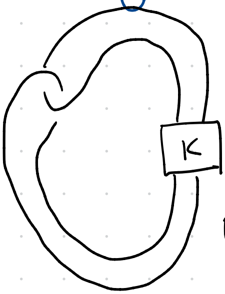
But, in general this will not be a minimal genus such surface

Example: $K \# -K$

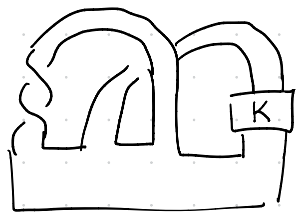
$$g(K\#-K) = 2g(K)$$

$$g_4(\underbrace{K\#-K}_{\text{slice!}}) = 0$$

Finding slice disks / constructing slice knots



$Wh(K)$ =



Seifert form

$$V = \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}$$

or

$$\Delta_{P(K)}(t) = \Delta_K(t^w)\Delta_{P(K)}(t)$$

Recall: $\Delta_{Wh(K)}(t) = 1$

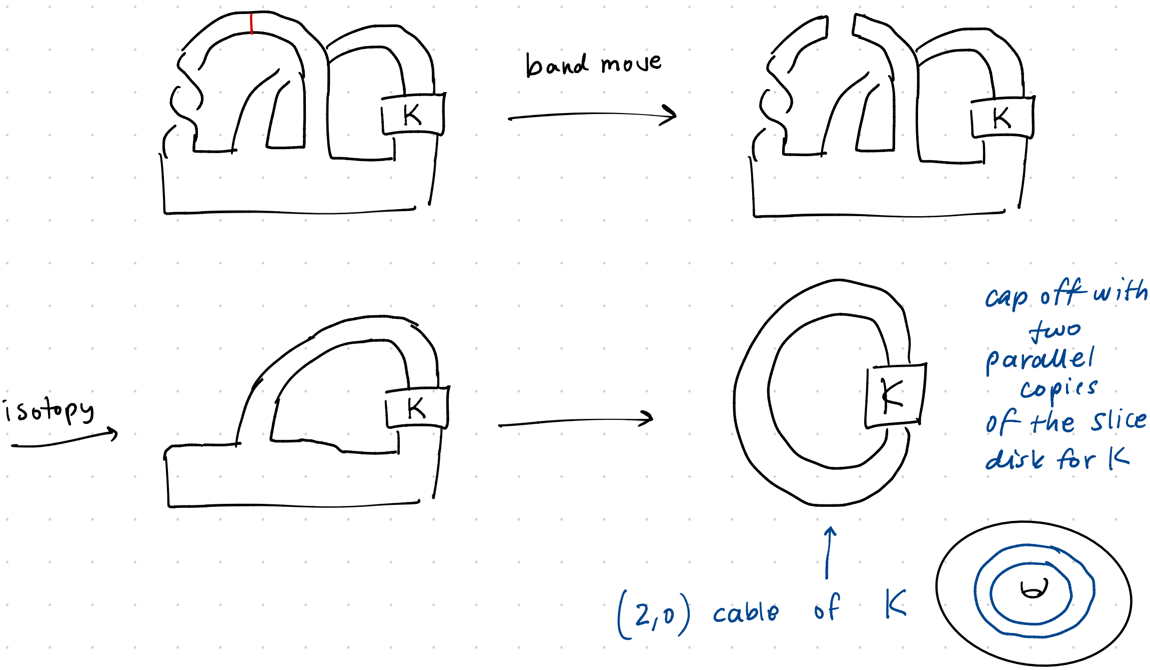
Big Theorem: (Freedman)

> If $\Delta_J(t) = 1$, then J is topologically slice

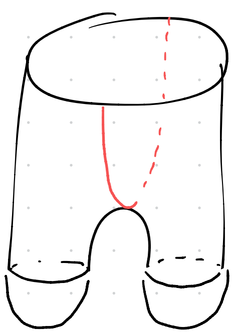
It follows from Freedman's Thm that $Wh(K)$ is topologically slice $\forall K$

Recall: If K is smoothly slice, then $Wh(K)$ is smoothly slice. i.e. $K \sim U \Longrightarrow Wh(K) \sim Wh(U) = U$

One way to explicitly see that K slice $\Rightarrow$ $Wh(K)$ slice is

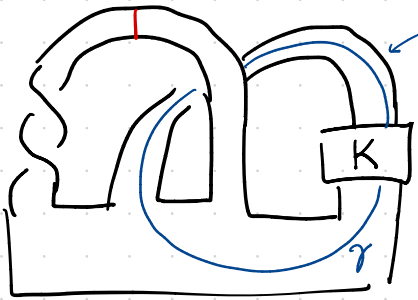


Schematic Picture:



(*) Important that these parallel copies of K linked zero times.

Another way to see if K smoothly slice $\Longrightarrow$ $Wh(K)$ smoothly slice is



Slice knot sitting on Seifert surface with $lk(\gamma, \gamma^+) = 0$

positive push-off

this is called the surface framing

Cut F along γ and cap off the resulting boundary components with two parallel copies of slice disk for K

Abstractly: