8803

Week 4 Notes

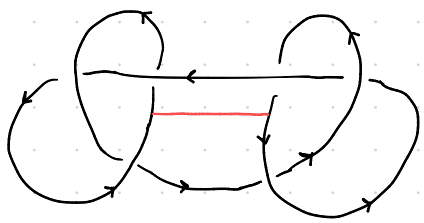
Monday pg 2

Wednesday pg 10

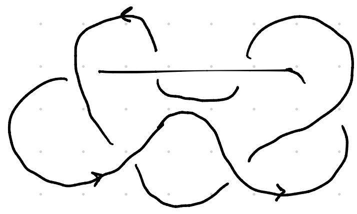
Remarks:

- Exercises (HW) due Friday
- Dr Hom's office hours: 1-2 pm M 1/28
  2-3 pm F 1/31
- For pretzel knot exercise, $P(-n,n,k)$ - n is odd!

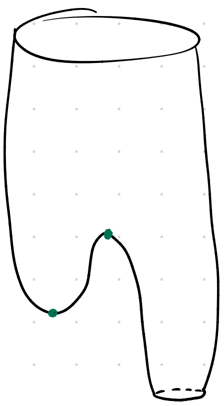
Last time: $K_0 \sim K_1 \Rightarrow P(K_0) \sim P(K_1)$ "satellite the concordance"



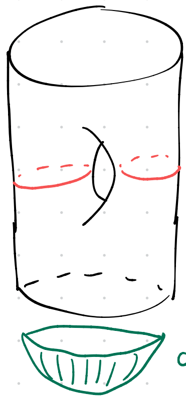
band move →



Remark:
- maintain orientation w/ band move
- make sure you're not creating genus



← this concordance has a saddle coming from the band move
(2 critical points)

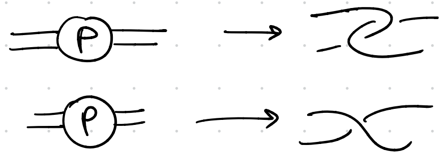


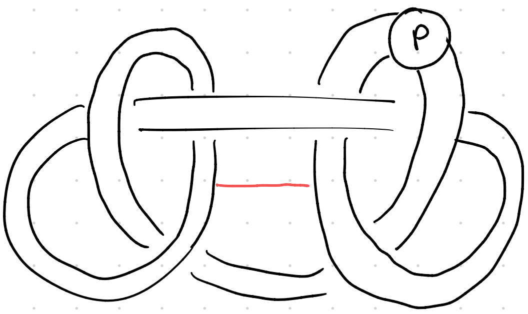
← creating 2 components

← merging back together

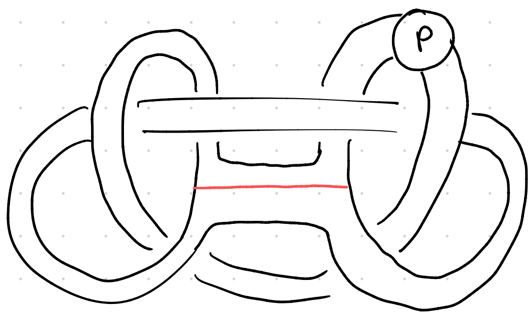
cap off disks at end

Example:

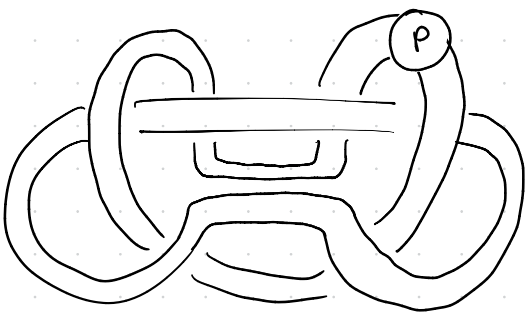




first band move →

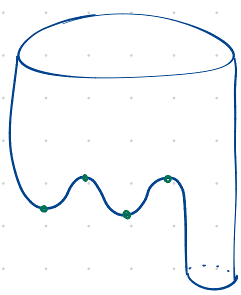


second band move →



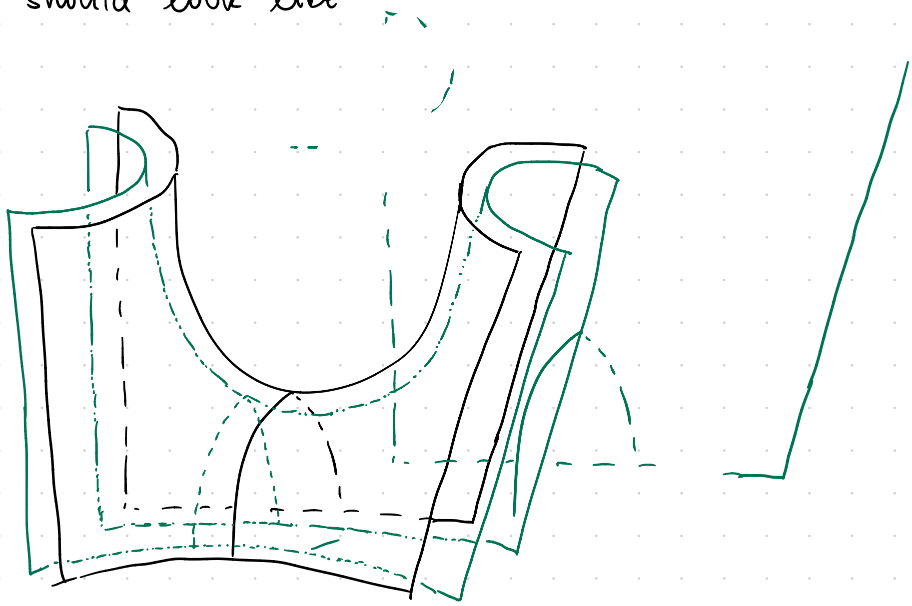
Rmk:
- regardless of winding no. can check that these are oriented

Abstractly our concordance would look like



4 critical points

Picture in your head should look like



Upshot: Given pattern you get a well-defined map of sets

$$P : \mathcal{C} \longrightarrow \mathcal{C}$$

in general this is not a homomorphism. (except id, 0-map, map induced by orientation reversal)

Remark: $\{K, mK, K^r, mK^r\}$ — mirror

could be 1, 2, or 4 different knots.

(check that it can't be 3)

Similarly, also might be 1, 2, or 4 different concordance classes

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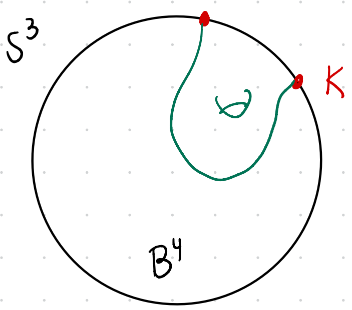
Slice genus / 4 ball genus

being slice is same as having 4ball genus zero.

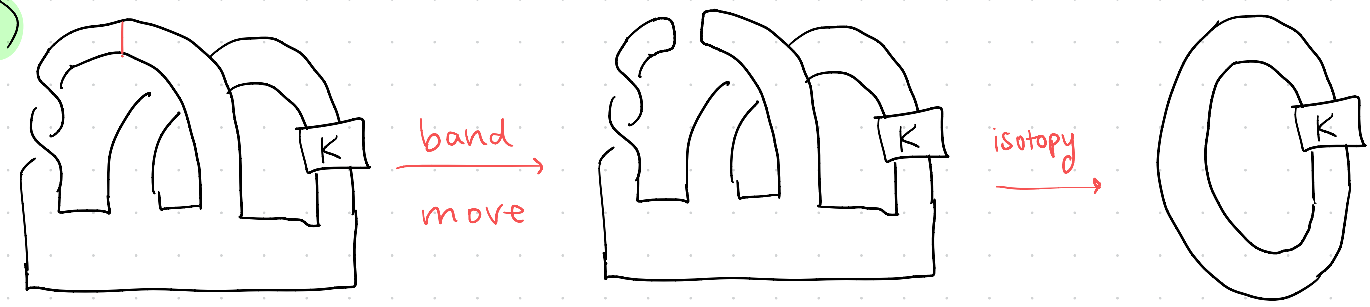


$$\left|\frac{\sigma(K)}{2}\right| \leq g_4^{TOP}(K)$$

$$g_4^{TOP}(K) \leq g_4^{smooth}(K) \leq g(K)$$

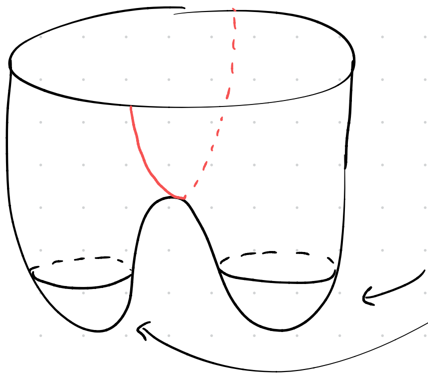
Example: K smoothly slice $\Longrightarrow$ wh(K) smoothly slice

1)



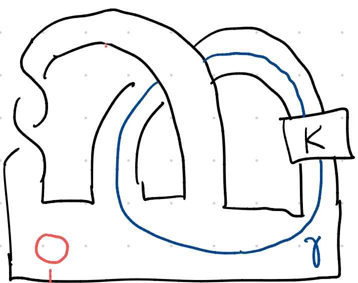
cap off with two parallel copies of slice disk for K

$\ast$ important that linking no. is zero



slice disks for K

2) Alternatively,



γ can't look like this

cut along γ and cap off each boundary component with a slice disk

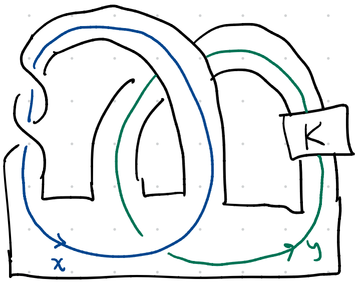
$lk(\gamma, \gamma^+) = 0$ so that the two parallel copies are disjoint

slice disks for K

<u>Key point:</u> If a genus 1 Seifert surface for K contains a homologically essential slice knot (γ in the picture) with surface framing zero, then K is slice

Recall that an algebraically slice knot K has a metabolic Seifert form. A geometric realization of a basis for the metabolizer is called a derivative of K

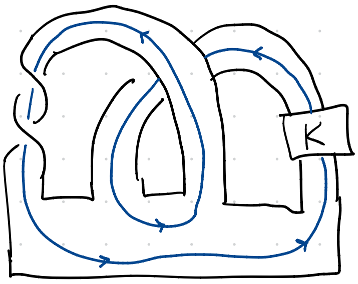
Example:



$$V = \begin{array}{cc} & \begin{array}{cc} x^+ & y^+ \end{array} \\ \begin{array}{c} x^- \\ y^- \end{array} & \begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix} \end{array}$$

y is an example of a derivative

Another example of a derivative in this picture:



$x + y$ (positive addition b|c it respected orientation)

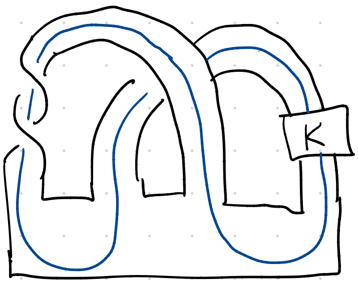
is also a derivative.

Need to check that the surfrace framing is zero.

(The Reidemeister 1 move  gives a -1

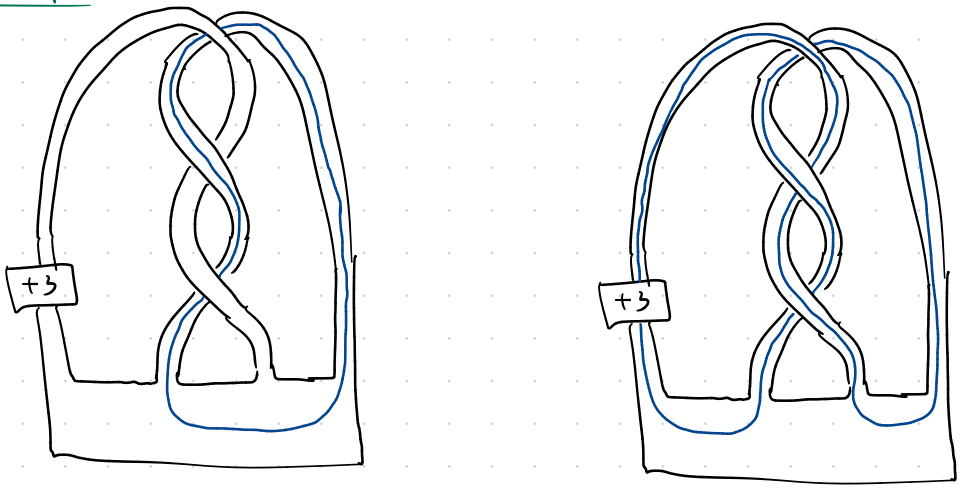
and knot gives a $+1$)

Note:



represents the homology class $x - y$.

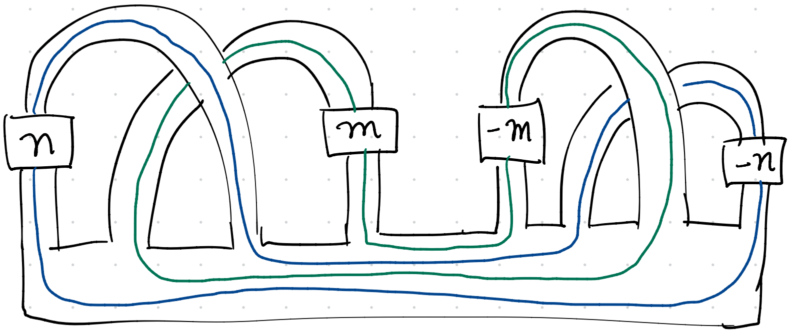
Example:



derivative

also a derivative

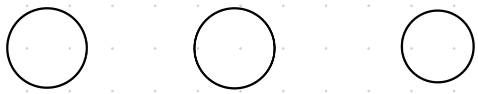
Genus Two Example:



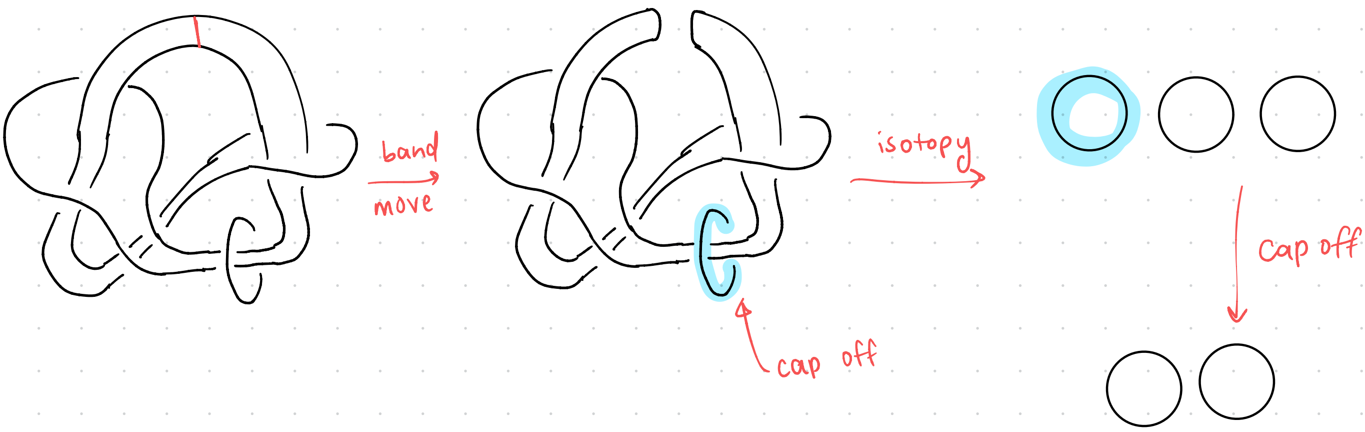
Now we need 2 curves.

Def'n: An n-component link is strongly slice if L bounds n disjoint disks in B^4

Ex:



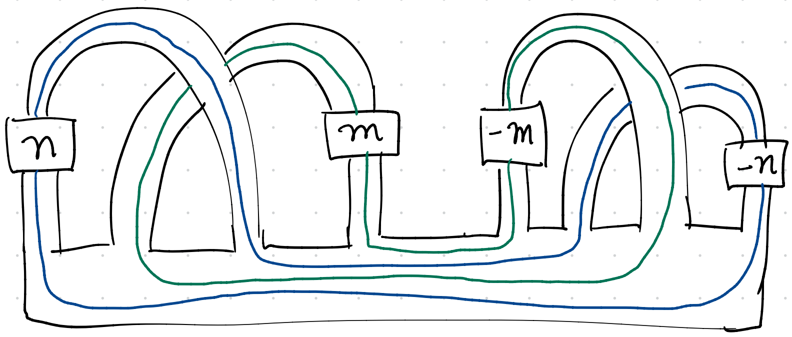
Ex:

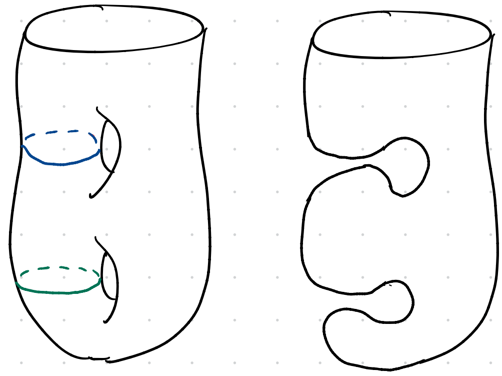


K slice $\Rightarrow$ K algebraically slice

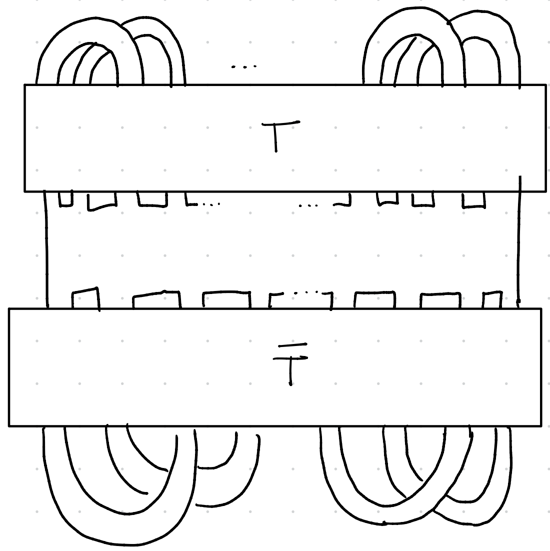
K slice $\Leftarrow$ K algebraically slice with a Seifert surface of genus g with g component slice link representing basis for metabolizer

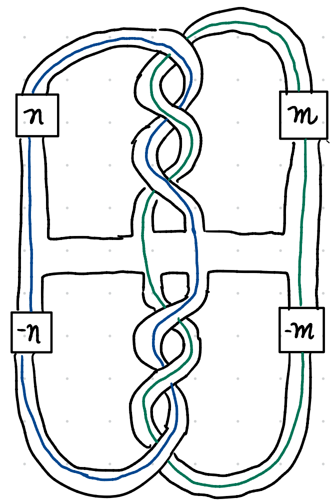
Example:





Example:





Remark:

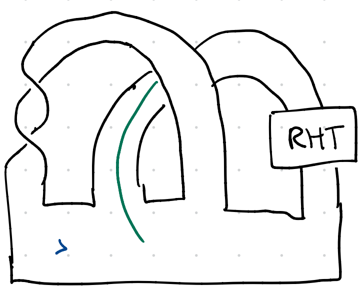
In higher odd dimensions $S^{2n+1} \hookrightarrow S^{2n+3}$
$n \geq 1$

Levine's proof that $C \cong \mathcal{G}$ algebraic concordance group relied on showing that any Seifert "surface" for a slice knot, $\exists$ a geometric realization of a basis for metabolizer by a slice link.

In classical dim, early proofs that $\mathcal{C} \twoheadrightarrow \mathcal{G}$ had nontrivial kernel relied on showing that if K is slice then certain signatures of derivatives for K must vanish

Example: Algebraically slice (in fact, topologically slice) knot that is not smoothly slice

$Wh(RHT)$



- Can we use Donaldson's work to show $Wh(RHT)$ not smoothly slice?

- Alternatively, can we use Ozváth-Szabó τ-invariant

$\tau(Wh(RHT)) = 1$ $\qquad$ $\tau : \mathcal{C}_{\text{smooth}} \to \mathbb{Z}$

Q: If K slice, does every Seifert surface for K admit a derivative that is a slice link?

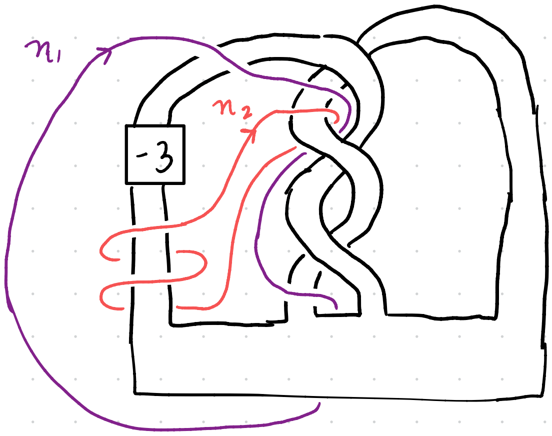
A: No.

Kauffman's conjecture:

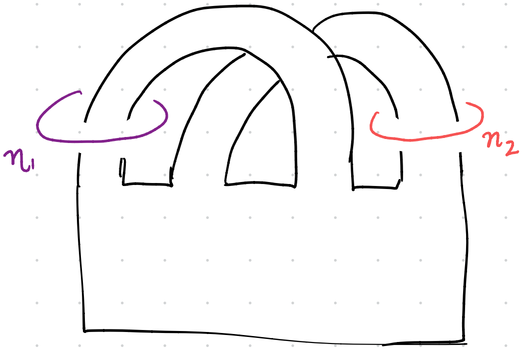
If K is a slice knot with a genus 1 seifert surface F, then $\exists$ an essential simple closed curve d on F s.t.

- $lk(d, d^+) = 0$
- d is a slice knot

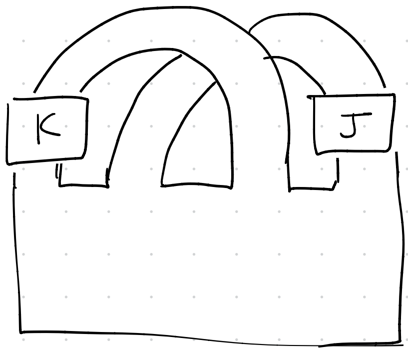
disproved by Cochran-Davis 2013.



$R_{\eta_1,\eta_2}(K,-K)$



$Q_{\eta_1,\eta_2}(K,J)$



1.) $R_{\eta_1,\eta_2}(K,-K)$ smoothly slice $\forall$ K (genus 1 slice knot)

2.) Identify derivatives d_1, d_2

$\mathbb{Z}_2$ valued concordance invariants

3.) Show that if $Arf(K) \neq 0$, then $Arf(d_1) \neq 0$

$Arf(d_2) \neq 0$

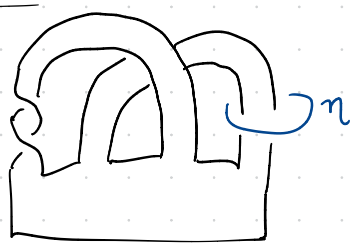
Wednesday

Reference:

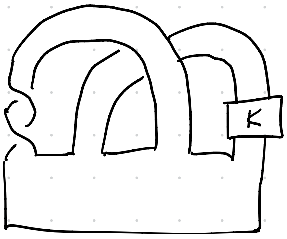
Slice Knots and Knot Concordance, Arunima Ray

Last time:

Ex:



$P_\eta(K) =$



"infection along a curve"

Two component link in S^3 and 1 component is an unknot gives rise to a satellite.

↳ complement of blue curve is a solid torus and that gives your pattern

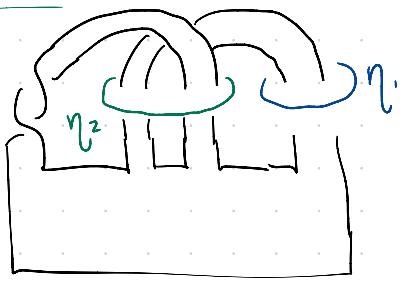
Remark:

Equivalently $P_\eta(K)$ can be described as the union of complement of η with $S^3 - \nu(K)$ where

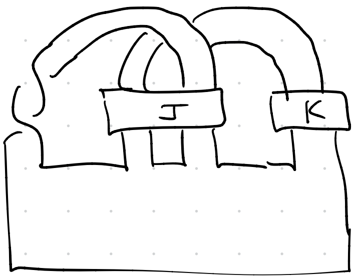
meridian of $\eta \longrightarrow$ 0-framed longitude of K

0-framed longitude of $\eta \longrightarrow$ meridian of K

Example:

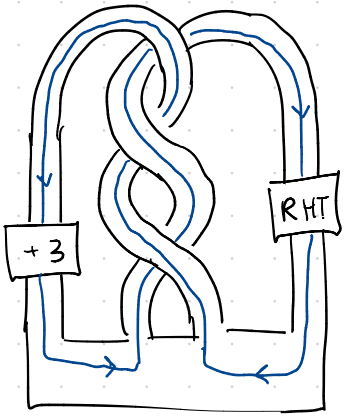


$P_{\eta_1,\eta_2}(K,J) =$



Using curves on Seifert surface to build slice disk

Ex: $K =$



Surface framing 0 ✓

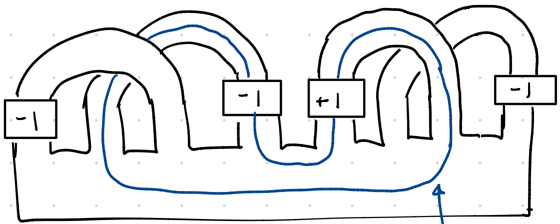
LHT # RHT slice ✓

blue curve is a derivative of K ✓

$\Longrightarrow$ K is slice

Not introducing any twisting

Ex: $K =$



$= RHT \# 4_1$ (Exercise to verify this)

figure 8

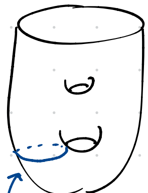
0-framed unknot sitting on surface

$\sigma(K) = -2 \Longrightarrow K$ not top slice

$\left|\frac{\sigma(K)}{2}\right| \leq g_4^{TOP}(K)$

$1 \leq g_4^{TOP}(K) = 1$

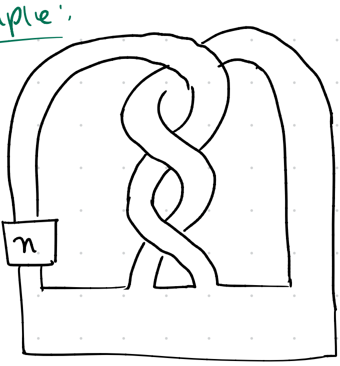
$1 \leq g_4^{smooth}(K) = 1$



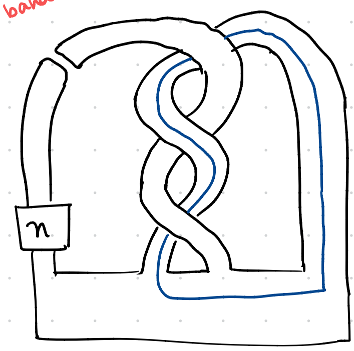
cutting along will reduced genus on smooth gen. 2 sfc

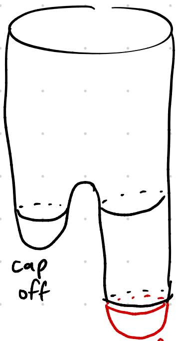
(if a separating curve, can do similar argument of seeing which part of the sfc it lobs off)

Example:



after an oriented band move

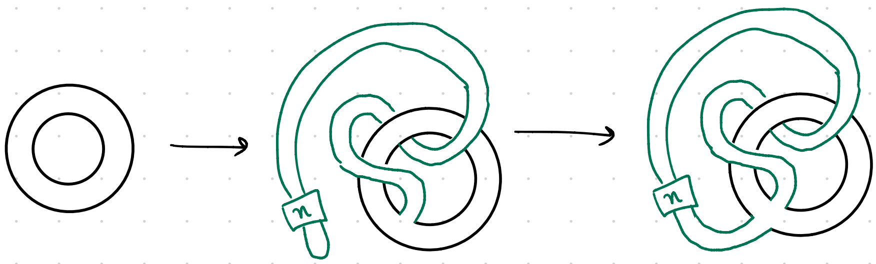




cap off

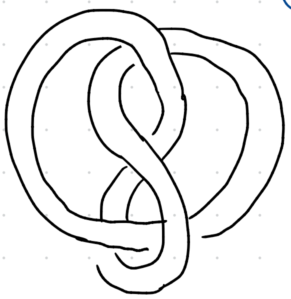
cap off

Movie:



Exercise: $g_4^{\text{smooth}}\Big(\underbrace{(2,1)\text{-cable of } 4_1}\Big) \le 1$

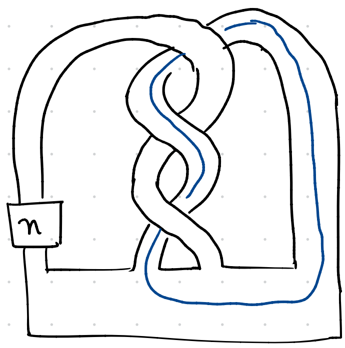
for a long time this was known to be not ribbon



Recently proved to be not slice

Defn: A Seifert surface is excellent if it admits a derivative that is a trivial link

Ex:



is excellent

Exercise: $J \# -J$ admits an excellent Seifert surface
(mirror)

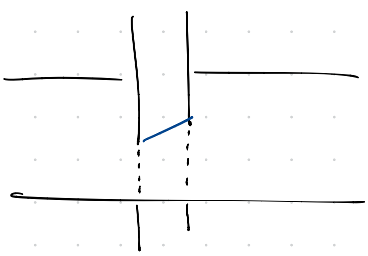
Note: If K admits an excellent Seifert surface, then K is smoothly slice (surger along unlink)

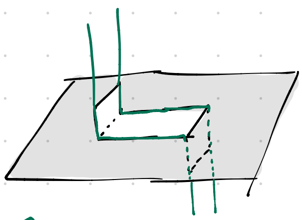
Proposition (Cochran–Davis)

Every slice knot is ribbon $\Longleftrightarrow$ Every slice knot has an excellent Seifert surface

Proof:

($\Longrightarrow$) Need to show a ribbon knot admits an excellent Seifert sfc.





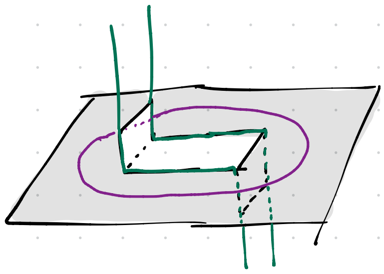
desingularize the ribbon singularity

increases the genus each time

Now we've gone from an immersed ribbon disk $\Longrightarrow$ Seifert surface

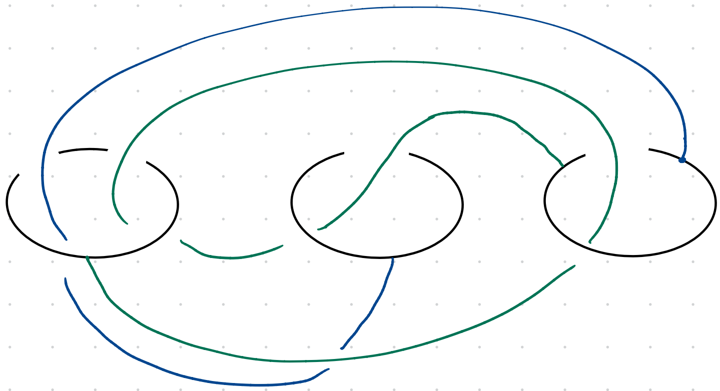
Claim: This Seifert surface is excellent

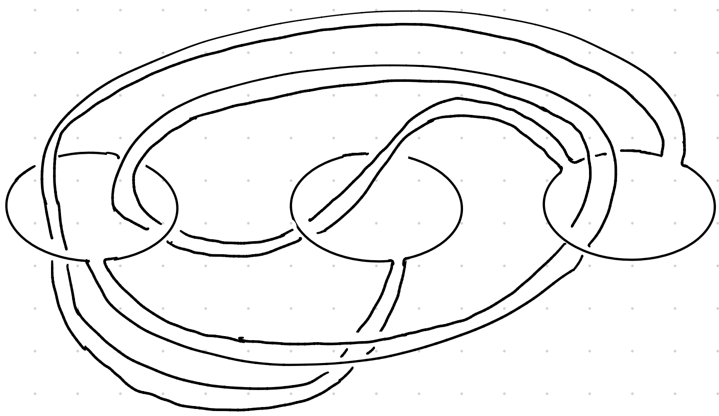
Proof:



($\Longleftarrow$) Need to show that if K has an excellent Seifert surface, then K is ribbon.

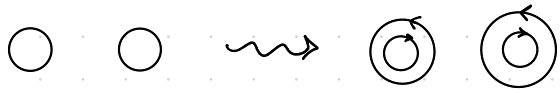
Recall: a ribbon disk can be described via an n-component unlink (minimal) and $n-1$ bands. (saddles)



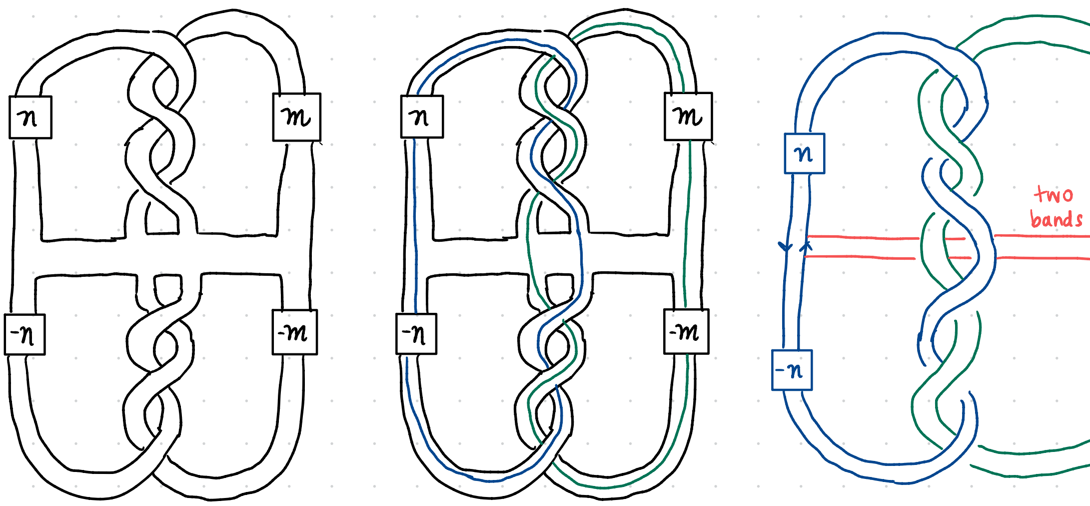


Suppose K has an excellent genus g Seifert surface F with associated g component unlink L

Then take two anti-parallel copies of L and attach $2g-1$ bands to obtain immersed ribbon disk for K



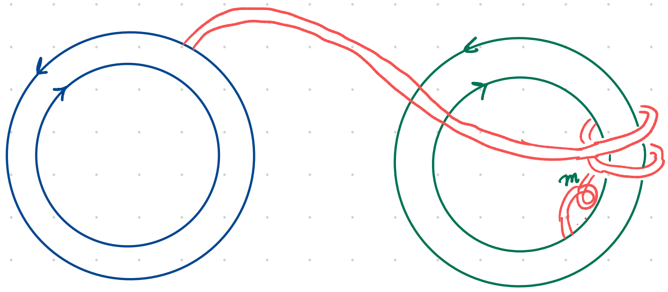
Example:



4 component unlink

3 bands

after isotopy:



Exercise: Show that in general, these constructions are not inverses to one another.

So far:

- explicit constructions of slice knots
- found slice obstructions via concordance invariants
  - Arf invariant
  - Fox-Milnor
  - signature
  - Levine-Tristram signatures

Other concordance invariants:

- Rasmussen s-invariant coming from Lee perturbation of Khovanov homology
- Osváth-Szabó τ-invariant coming from knot Floer homology
- many more coming from variations of Heegaard Floer and Khovanov homology

Note: Concordance invariants coming from Khovanov homology and Floer homology are invariants of smooth knot concordance, in contrast to invariants coming from Seifert form, which are invariants of topological knot concordance

$$\mathcal{C}_{smooth} \twoheadrightarrow \mathcal{C}_{TOP}$$

surjects

$$\sigma : \mathcal{C}_{TOP} \longrightarrow 2\mathbb{Z}$$

$\tau, s : \mathcal{C}_{smooth} \longrightarrow \mathbb{Z}$ don't factor through $\mathcal{C}_{TOP}$

Next theme: 4-mfds can be useful for studying knot concordance, and vice versa

Knot trace

Defn: the n-trace of a knot $K \subset S^3$ is

$$X_n(K) := B^4 \cup \begin{array}{l} n\text{-framed 2-handle} \\ \text{attached along } K \subset \partial B^4 \end{array}$$

$$= B^4 \cup_\varphi (D^2 \times D^2)$$

core is $D^2 \times \{0\}$

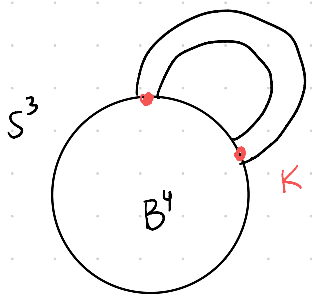
$\partial(D^2 \times D^2)$
$= (\partial D^2 \times D^2) \amalg (D^2 \times \partial D^2)$

$$\varphi : \partial D^2 \times D^2 \longrightarrow \nu(K)$$

$$S^1 \times \{x\} \longmapsto n\text{-framed longitude of } K$$

$x \in \partial D^2$

Note: the trace $X_n(K)$ has boundary



$$(S^3 - \nu(K)) \cup (D^2 \times S^1)$$

Dehn surgery

φ specifies how the solid torus was glued in.

Exercise: use gluing map ϕ to conclude that $\partial(X_n(K)) = S^3_n(K)$