Week 4

8803

Monday pg 2

Wednesday pg 11

Last time:

$X_n(K) = B^4 \cup$ n-framed 2-handle attached along K

$= B^4 \cup_\varphi (D^2 \times D^2)$

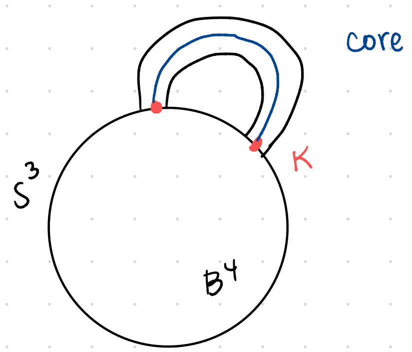
$\varphi : \partial D^2 \times D^2 \longrightarrow \nu(K)$

$S^1 \times \{x\} \longmapsto$ n-framed longitude of K

$x \in \partial D^2$

core of 2-handle is $D^2 \times \{0\}$

$\partial(D^2 \times D^2) = \partial D^2 \times D^2 \amalg D^2 \times \partial D^2$



Note: $\partial(X_n(K)) = S^3_n(K)$

Example:

$X_0(U) = S^2 \times D^2$

top. embedding where image has a collared nbhd

Trace Embedding Lemma

The 0-trace $X_0(K)$ admits a smooth (resp. locally collared) embedding into S^4 iff K is smoothly (resp. topologically) slice.

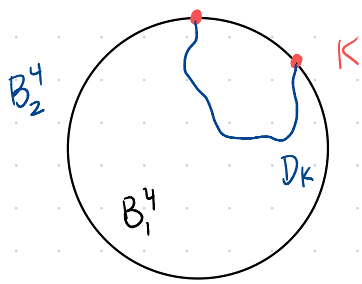
Proof:

($\Leftarrow$) Suppose K is slice

Consider $S^4 = B^4_1 \cup_{S^3} B^4_2$

$X_0(K) \cong B^4_2 \cup \nu(D_K)$

is the desired embedding

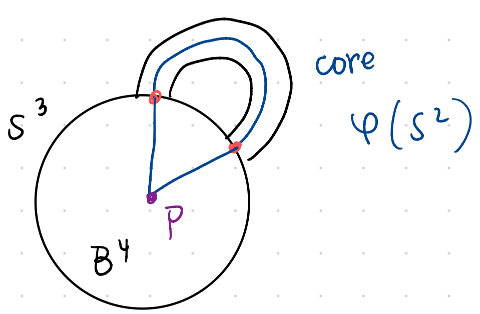


Exercise: This is a 0-framed 2-handle

($\Rightarrow$) $\varphi : S^2 \longrightarrow X_0(K)$

Not smooth!

$\varphi(S^2) =$ core of the 2-handle $\cup$ cone on K



φ is smooth away from the cone point P

$i : X_0(K) \longrightarrow S^4$ embedding

$i \circ \varphi : S^2 \longrightarrow S^4$ smooth (resp. locally flat) away from $i(p)$

$W := S^4 - \nu(i(p)) \cong B^4$

Consider $i \circ \varphi \big|_{S^2 - \varphi^{-1}(p)}$ is smooth (resp. top. locally flat)

embedding of D^2 into $W \cong B^4$

///

Corollary

$X_0(K)$ smoothly (resp. locally collared) embeds into $\mathbb{R}^4$

$\iff$

K is smoothly (resp. top.) slice

Proof: Exercise

Goal: Use knots to construct an exotic pair of 4-mfds

Defn: M_1, M_2 smooth 4-mfds

M_1, M_2 are an *exotic pair* if they are homeomorphic but not diffeomorphic.

Thm (Freedman–Quinn)

Let M be a connected, non-compact 4-mfd. If desired, fix a smooth structure on any collection of connected cmpnts of the boundary. Then $\exists$ a smooth structure on M extending the given smooth structure on (a subset of) ∂M

Construction: (Gompf–Stipsicz Exercise 9.4.23)

Let K be a topologically slice knot that is not smoothly slice. Then can construct a smooth manifold $\mathcal{R}$ that is homeomorphic but not diffeomorphic to $\mathbb{R}^4_{std}$

- Let K be topologically slice but not smoothly slice (eg. $Wh(RHT)$)

- $\exists$ a locally collared embedding of the trace into $\mathbb{R}^4$

$$i : X_0(K) \longrightarrow \mathbb{R}^4$$

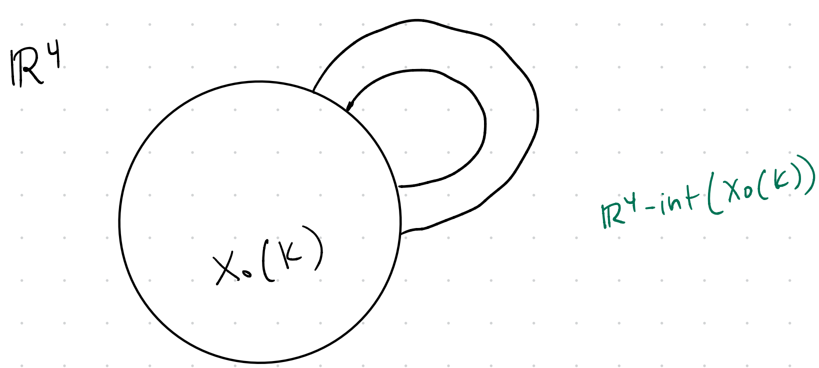
by corollary to Trace Embedding Lemma

- $i(X_0(K))$ inherits a smooth structure from $X_0(K)$

- Since i is locally collared, $\mathbb{R}^4 - \mathrm{int}(X_0(K))$ is a manifold.

  Also, it is connected and non-compact.

- Freedman-Quinn $\Longrightarrow$ Extend smooth structure on $i(\partial X_0(K))$ to the rest of $\mathbb{R}^4 - \mathrm{int}(X_0(K))$ giving a smooth mfd $\mathcal{R}$ that is homeomorphic to $\mathbb{R}^4$



<u>Claim:</u> $\mathcal{R}$ is not diffeomorphic to $\mathbb{R}^4_{std}$

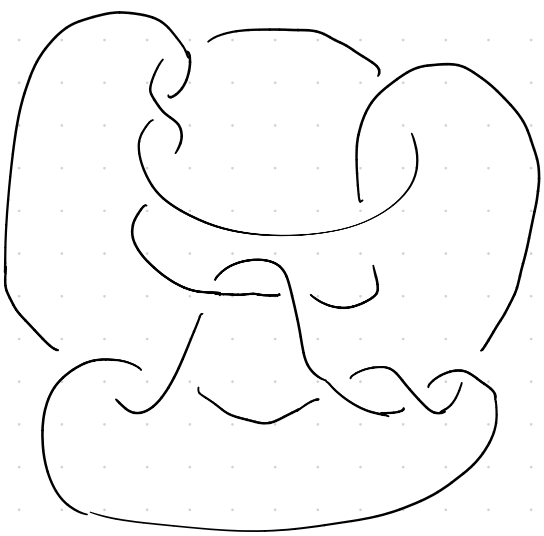
Proof of Claim: Suppose it were. Then by construction, we have a smooth embedding

$$X_0 \longrightarrow \mathcal{R} \cong_{\text{diffeo}} \mathbb{R}^4$$

Corollary to Trace Embedding Lemma $\Longrightarrow$ K is smoothly slice $\otimes$

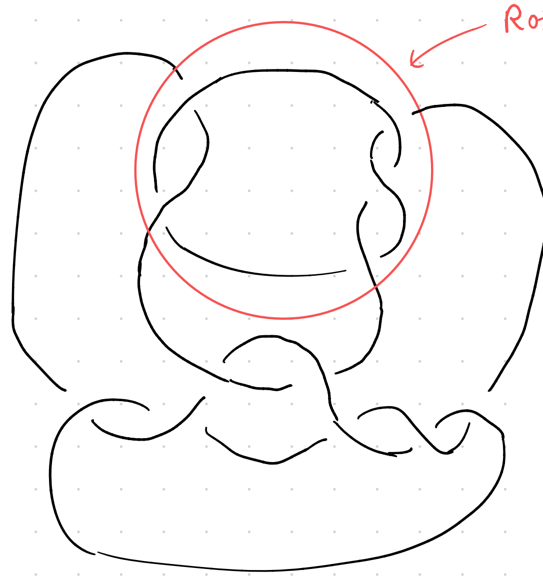
///

We can also use 4-mfds to prove results about knots:



Conway Knot

C



Rotate by 180° then the ends match up and give Conway Knot

Mutant of the Conway Knot

Kinoshita-Terasaka knot

KT

Exercise:

1. $\Delta_{KT}(t) = \Delta_C(t) = 1$

2. KT is ribbon

Q: Is the Conway knot smoothly slice?

All known smooth concordance invariants vanish on C.

A: No (Picarillo)

Proof (sketch)

- Build a knot J with $X_0(J) \cong X_0(C)$
- Show that J is not smoothly slice using $s(J) \neq 0$

T.E.L implies $X_0(J)$ does not smoothly embed into S^4

Hence $X_0(C)$ does not smoothly embed in S^4, hence

C is not smoothly slice. ///

s is not a 0-trace invariant.

In particular, $s(C) = 0$

How do you find J?

$$X_0(C) = B^4 \cup (2\text{-}h)$$
$$= B^4 \cup (1\text{-}h) \cup 2(2\text{-}h)$$

add a cancelling pair s.t. the 1-h can cancel either of the 2-h's

(then this is the trace of 2 knots)

Khovanov Homology and the s-invariant

Melissa Zhang Notes on Khovanov homology

Outline:
- Kauffman bracket
- Jones Polynomial
- Khovanov homology
- Lee perturbation
- Rasmussen s-invariant

Defn: the Kauffman bracket $\langle D \rangle$ of a link Diagram D is the Laurent polynomial in q defined recursively by

1.) $\langle \times \rangle = \langle \asymp \rangle - q \langle)(\rangle$

2.) $\langle L \amalg \underset{\text{unknot}}{O} \rangle = (q+q^{-1}) \langle L \rangle$

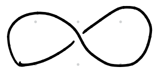
3.) $\langle \phi \rangle = 1$

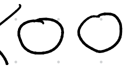
Example: 1. $D_1 = \bigcirc$ $\qquad \langle D_1 \rangle = q + q^{-1} = \langle \phi \amalg O \rangle$

2. $D_2 = \bigcirc \; \bigcirc$ $\qquad \langle D_2 \rangle = (q+q^{-1})^2$

3. $D = \underbrace{O \cdots O}_{n}$ $\qquad \langle D \rangle = (q+q^{-1})^n$

Example:

4. $D_4 =$ 

$\langle D_4 \rangle = \langle$  $\rangle - q \langle$  $\rangle$

$= (q + q^{-1}) - q(q + q^{-1})^2$

$= -q^3 - q$

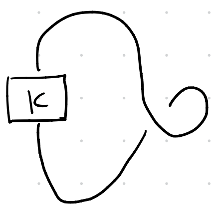
isotopic to unknot

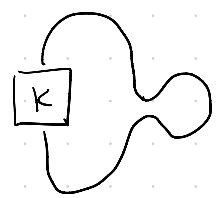
but has different Kauffman bracket.

Note:

Kauffman bracket is not a link invariant

Exercise:

$D =$ 

$D' =$ 

Q: How does the Kauffman bracket change? A: $\langle D \rangle = -q^2 \langle D' \rangle$

Let D be a diagram with n_- negative and n_+ positive crossings.

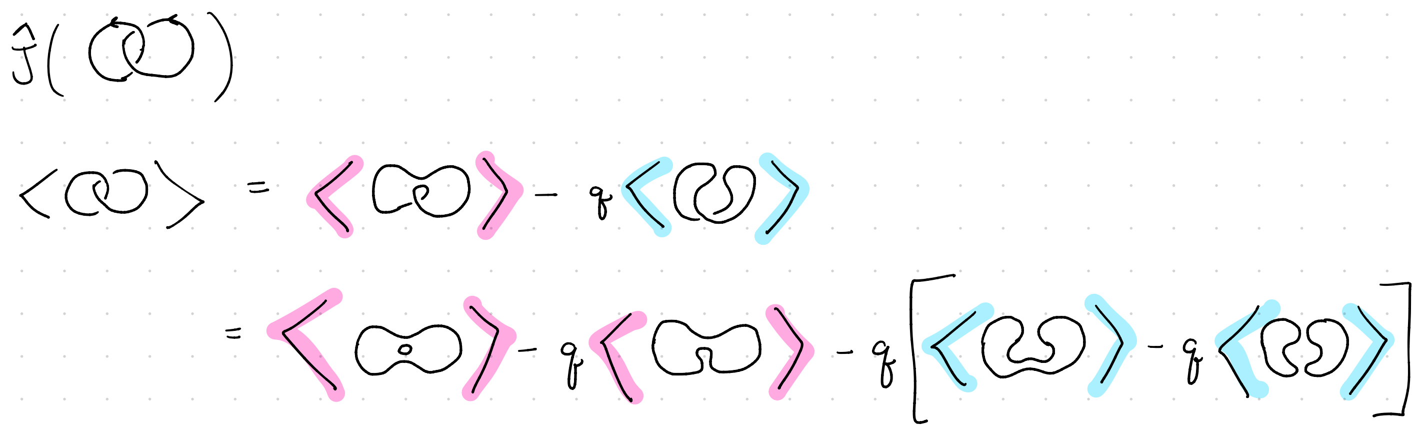
Defn: The (unnormalized) Jones polynomial of D is

$$\hat{J}(D) = (-1)^{n_-} + q^{n_+ - 2n_-} \langle D \rangle$$

Exercise: $\hat{J}(D)$ is a link invariant

Def'n: The Jones polynomial is $J(D) = \dfrac{\hat{J}(D)}{q+q^{-1}}$

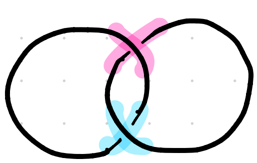
Example:



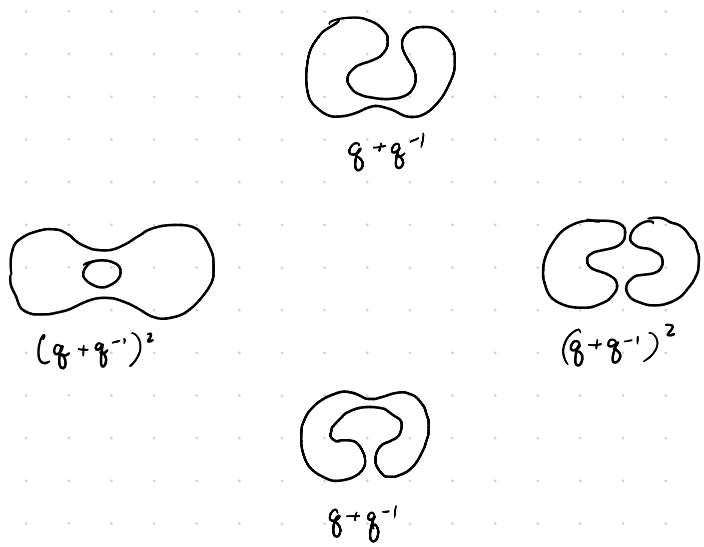
$= (q+q^{-1})^2 - q(q-q^{-1}) - q\left((q+q^{-1}) - q(q+q^{-1})^2\right)$

$= q^4 + q^2 + 1 + q^{-2}$

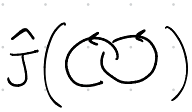
Can resolve



simultaneously:



To recover (unnormalized) Jones polynomial:



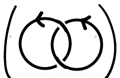
$= (-1)^0 q^2 (q^4 + q^2 + 1 + q^{-2})$

$n_+ = 2$

$n_- = 0$

$= q^6 + q^4 + q^2 + 1$

Last time:

$$\hat{J}\Big($$$$\Big) = (-1)^0 q^2\left((q+q^{-1})^2 - 2q(q+q^{-1}) - q(q+q^{-1})^2\right)$$

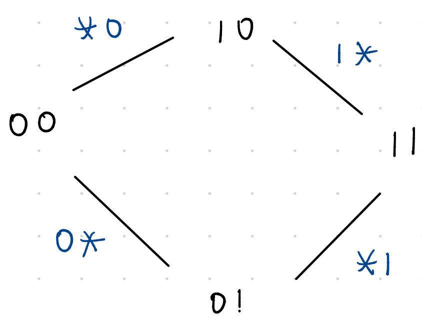
Goal: look at s-invariant because it's a powerful invariant

State-Sum formula for $\hat{J}(D)$

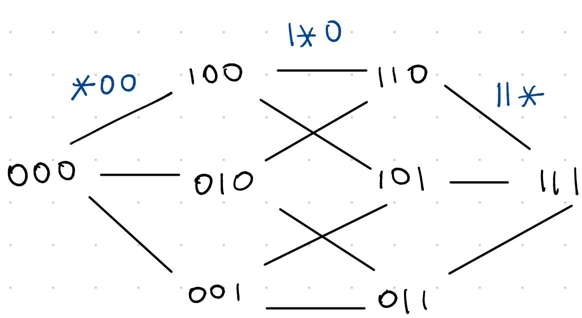
Let D be a link diagram with n crossings.

Consider n-dim hypercube $\{0,1\}^n$ with edges between vertices differing in exactly one place

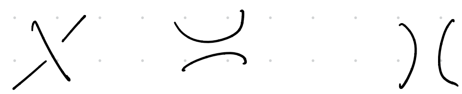
Ex: $n=2$



Ex: $n=3$



$\alpha \in \{0,1\}^n$



0 resolution 1 resolution

Γ_α = associated collection of circles

r_α = # of 1's in α

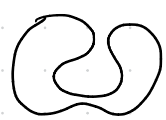
k_α = # of circles in Γ_α

<u>Claim:</u> $\hat{J}(D) = (-1)^{n_-} q^{n_+ - 2n_-} \cdot \sum_{\alpha \in \{0,1\}^n} (-q)^{r_\alpha} (q+q^{-1})^{k_\alpha}$

<u>Proof:</u> Exercise

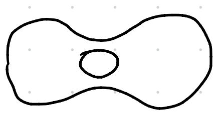
$n_- = 0$
$n_+ = 2$

10



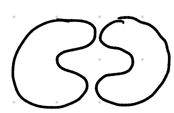
$q + q^{-1}$

00



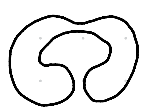
$(q+q^{-1})^2$

11



$(q+q^{-1})^2$

01



$q+q^{-1}$

<u>Goal:</u> diagram D $\rightsquigarrow$ bigraded chain complex $CKh(D)$

s.t. • (co-) homology of $CKh(D)$ is an invariant

• graded Euler characteristic is Jones polynomial

$$\chi_q(CKh(D)) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \operatorname{rk} H^{i,j}(CKh(D)) = \hat{J}(D)$$

$V = \mathbb{Z}v_+ \oplus \mathbb{Z}v_-$ bigraded $\mathbb{Z}$-module

$v_\pm$ bigrading $(0, \pm 1)$

<u>Note:</u> $\chi_q(V) = q + q^{-1}$

<u>Notation:</u> $C = \bigoplus_{i,j} C_{i,j}$

<u>Note:</u> also have bigrading (h, q) instead of (i,j)

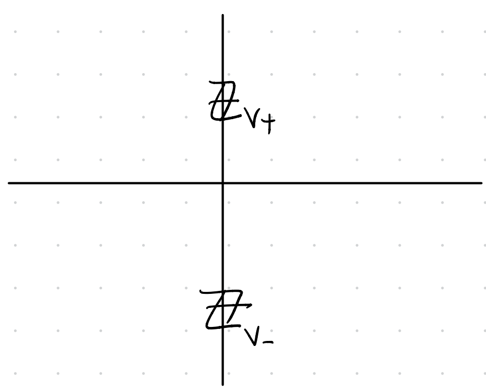
$$C[n]\{m\}_{i,j} = C_{i+n,\, j+m}$$

$[n]$ ↓ shift in 1st grading

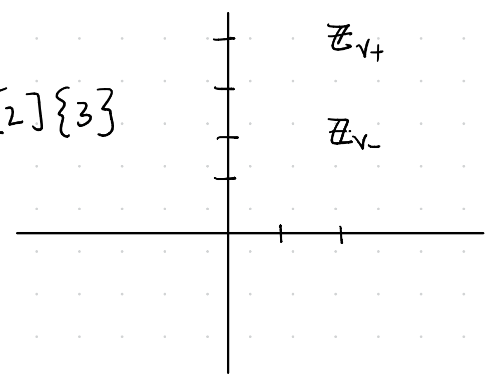
$\{m\}$ ↓ shift in 2nd grading

<u>Example:</u>

V

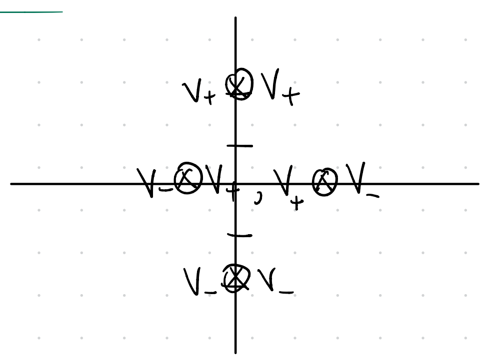


$V[2]\{3\}$

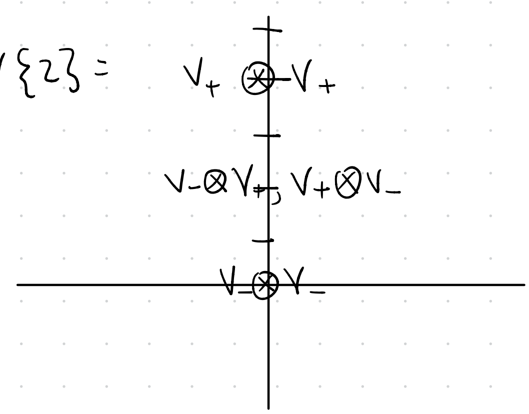


<u>Example:</u>

$V \otimes V =$



$V \otimes V\{2\} =$



Idea: Replace $(q+q^{-1})$ in state-sum formula for $\hat{J}(D)$ with V

$$\begin{tikzcd}[column sep=small]
 & \overset{\alpha = 10}{V[0]\{3\}} \arrow[dl, dash] \arrow[dr, dash] & \\
\underset{\alpha = 00}{V \otimes V[0]\{2\}} \arrow[dr, dash] & & \overset{11 = \alpha}{V \otimes V[0]\{4\}} \arrow[dl, dash] \\
 & \underset{\alpha = 01}{V[0]\{3\}} &
\end{tikzcd}$$

$h =$ 0   1   2   (homological grading)

no. of 1's in α

$\alpha \in \{0,1\}^n \qquad V_\alpha = V^{\otimes k_\alpha}[n_-]\{r_\alpha + n_+ - 2n_-\}$

k_α: no. of circles (no. of unknots)

bigrading (h,q)

q \ h	0	1	2	χ
6			•	$0-0+1 = 1$
4	•	••	••	$1-2+2 = 1$
2	••	••	•	$2-2+1 = 1$
0	•			$1-0+0 = 1$
				$q^6+q^4+q^2+1$

Exercise: The graded Euler characteristic gives $\hat{J}(D)$

Differential on $CKh(D)$

- happens along edges of hypercube
  - ↳ circles either merge or split along an edge

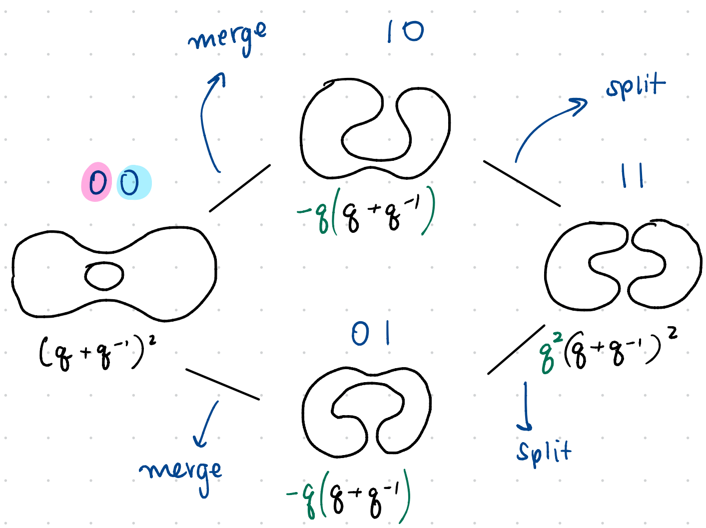
merge: two circles (so two copies of V) go to one

$$m : V \otimes V \longrightarrow V$$

$$v_+ \otimes v_+ \longmapsto v_+$$

$$v_- \otimes v_+ ,\ v_+ \otimes v_- \longmapsto v_-$$

$$v_- \otimes v_- \longmapsto 0$$



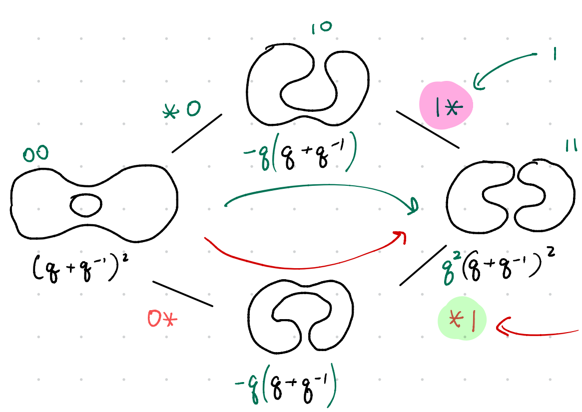
split:

$$\Delta : V \longrightarrow V \otimes V$$

$$v_+ \longmapsto v_+ \otimes v_- + v_- \otimes v_+$$

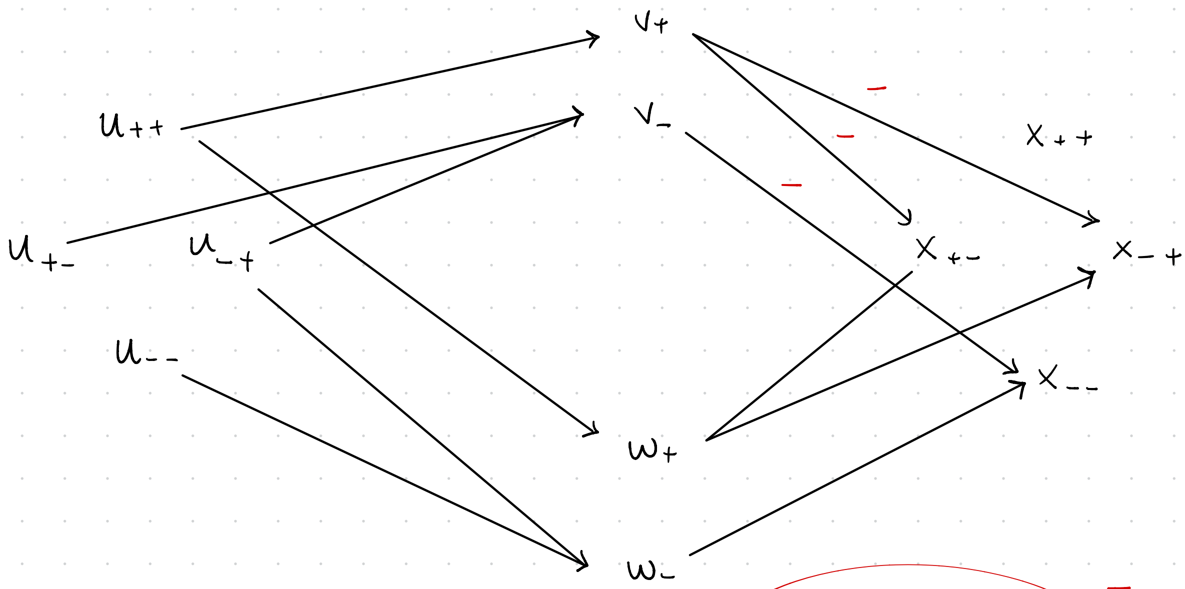
$$v_- \longmapsto v_- \otimes v_-$$

- differential is identity on passive circles along an edge
- $\pm$ signs needed along edge according to the no. of 1's before $*$ in edge label (needed so that $d^2 = 0$)



1 before $*$, so pick up a minus sign when you travel around top.

No sign picked up



$[x_{+-} + x_{-+}] = 0$

$q \backslash h$	0	1	2	χ
6			$[x_{++}]$	1
4			$[x_{+-}] = [-x_{-+}]$	1
2	$[u_{+-} = u_{-+}]$			1
0	$[u_{--}]$			1

$q^6 + q^4 + q^2 + 1$

Exercise:

Compute Khovanov homology of the trefoil

Theorem (Khovanov)

Khovanov homology is a link invariant

Proof: relies on showing invariants under Reidemeister moves

Main Idea:

Let C be a chain complex with subcomplex $A \in C$.

That is, A is a submodule of C and $dA \subseteq A$

Then we have a short exact sequence of chain complexes

$$0 \longrightarrow A \longrightarrow C \longrightarrow C/A \longrightarrow 0$$

Lemma

Let $0 \to A \to C \longrightarrow C/A \longrightarrow 0$ be a s.e.s. of chain complexes.

chain homotopy equiv.

1. If $A \simeq 0$, then $C \simeq C/A$
2. If $C/A \simeq 0$, then $A \simeq C$

Reidemeister invariance involves finding sub or quotient complexes that are null-homotopic

Q: How can Khovanov homology be used to study concordance?

Lee Perturbation:

merge: $m : V \otimes V \longrightarrow V$

$$v_+ \otimes v_+ \longmapsto v_+$$
$$v_- \otimes v_+,\ v_+ \otimes v_- \longmapsto v_-$$
$$v_- \otimes v_- \longmapsto 0 + v_+$$

split: $\Delta : V \longrightarrow V \otimes V$

$$v_+ \longmapsto v_+ \otimes v_- + v_- \otimes v_+$$
$$v_- \longmapsto v_- \otimes v_- + v_+ \otimes v_+$$

$$d_{Lee} = d_{Kh} + \Phi$$

purple terms

(h, q)

homological grading

quantum grading

But wait! d_{Lee} does not preserve the quantum grading

Filtered Chain Complexes

Let (C, d) be a chain complex. A filtration on (C,d) is a sequence subcomplexes

$$\dots \supseteq F_i \supseteq F_{i+1} \supseteq F_{i+2} \supseteq \dots$$

such that

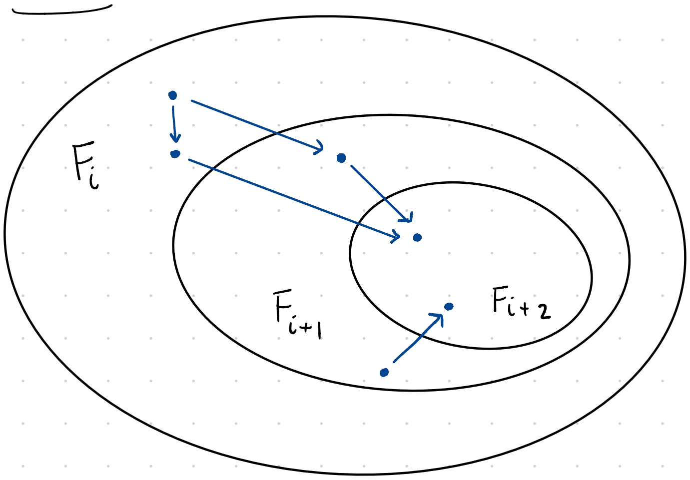
$$\bigcap_i F_i = \phi \qquad \bigcup_i F_i = C$$

Generally we will be interested in finite length filtrations where only finitely many F_i are not $\emptyset$ or C

i.e.

$$C = F_n \supseteq F_{n+1} \supseteq \dots \supseteq F_{n+k} = \emptyset$$

Idea:



Note: (CKh, d_{Lee}) is a filtered chain complex with

$$F_i = \bigoplus_{q \geq i} CKh_q$$