8803

Week 6

Monday pg 2

Wednesday pg 10

Last time:

Lee perturbation $\quad C_{Lee}(L)$

merge $\quad m' : V \otimes V \longrightarrow V\{\ell\}$

$$V_+ \otimes V_+ \longmapsto V_+$$

$$V_+ \otimes V_-,\ V_- \otimes V_+ \longmapsto V_-$$

$$V_- \otimes V_- \longmapsto V_+$$

perturbation

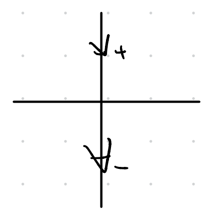
split $\quad \Delta' : V \longrightarrow V \otimes V\{\ell\}$

$$V_+ \longmapsto V_+ \otimes V_- + V_- \otimes V_+$$

$$V_- \longmapsto V_- \otimes V_- + V_+ \otimes V_+$$

$d_{Lee} = d_{Kh}$ + red terms

Exercise: Check $d_{Lee}^2 = 0$

$V =$ 

Note: Lee complex has a well-defined $\mathbb{Z}/4$ quantum grading

For our discussion of s, we will work with $\mathbb{Q}$-coefficients

(more generally works over a field of any characteristic)

$$V = \mathbb{Q}V_+ \oplus \mathbb{Q}V_-$$

Consider the change of basis

$$a = v_- + v_+$$
$$b = v_- - v_+$$

Note:

1. These elements don't have a well-defined quantum grading as they are not homogeneously graded

2. We can still consider the quantum filtration of a non-homogeneously graded element x

i.e. $$gr_q(x) = \max\{ i \mid x \in F^i(C_{Lee}) \}$$

$$a = v_- + v_+$$
$$b = v_- - v_+$$

Our things are filtered
$C \supseteq \dots \supseteq F_i \supseteq F_{i+1} \supseteq \dots \supseteq \emptyset$

and the quantum filtration of a homology class $[x]$:

$$gr_q(x) = \max\{ gr_q(y) \mid [x] = [y] \}$$

homologous

Exercise:

m': $a \otimes a \longmapsto 2a$

$a \otimes b,\ b \otimes a \longmapsto 0$

$b \otimes b \longmapsto -2b$

Δ': $a \longmapsto a \otimes a$

$b \longmapsto b \otimes b$

gives us how many components of L

Theorem (Lee)

The homology of an n-component link L is

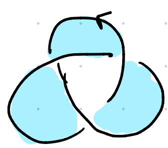
$$\mathrm{Lee}(L) \cong (\mathbb{Q} \oplus \mathbb{Q})^n$$

corresponding to the 2^n different orientations of L

Lee's canonical generators

Given a diagram D, $\exists$ a checkerboard coloring

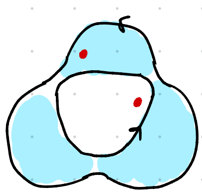
D



convention: leave the infinite region unshaded

Say knot is oriented, then $\exists$ an oriented resolution at each crossing.

D_0



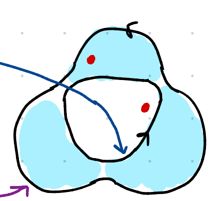
Checkerboard shading induces shading on D_0.

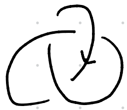
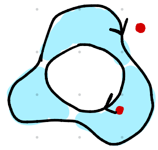
Now draw a dot to the left of any point on each circle.

If dot is in a shaded (resp. unshaded) region, label that circle with an a (resp b)

$S_0 = a \otimes b$

D_0



$D_{\bar{0}} =$   $s_o = b \otimes a$

opposite orientation

Exercise: $s_o \in \ker d_{Lee}$

Let K be a knot

$$s_{min}(K) = \min\{ gr_q([x]) \mid [x] \in Lee(K),\ [x] \neq 0 \}$$

$$s_{max}(K) = \max\{ gr_q([x]) \mid [x] \in Lee(K),\ [x] \neq 0 \}$$

Defn: (J. Rasmussen) (K is a knot)

$$s(K) = \frac{s_{max}(K) + s_{min}(K)}{2}$$

(*) Proposition (K is a knot)

$$s_{max} = s_{min} + 2, \quad \text{or equivalently} \quad s = s_{max} - 1 = s_{min} + 1$$

Exercise:

For a knot K, quantum gradings are all odd

Recall: $CLee(K)$ has a $\mathbb{Z}/4$ quantum grading, hence

$$CLee(K) \cong CLee_1(K) \oplus CLee_{-1}(K) \quad \text{where} \quad CLee_{\pm 1}(K)$$

is the summand in $\mathbb{Z}/4$ quantum grading ± 1

$\bar{o}$ is the opposite orientation

Lemma

Let D be a diagram for a knot K and o be an orientation. Then $s_o \pm s_{\bar{o}}$ are in two different $\mathbb{Z}/4$ quantum gradings of $CLee(D)$

Exercise: Proof of Lemma

Hence $s_o \pm s_{\bar{o}}$ generate the summands in $Lee(K) \cong Lee_1(K) \oplus Lee_{-1}(K)$

and hence

$$s_{max}(K) \ngeq s_{min}(K)$$

Lee homology supported in 2 dif. $\mathbb{Z}/4$ gradings. Lifting to filtration tells you $\exists$ two dif. generators

and

$$s_{min}(K) = gr_q([s_o]) = gr_q([s_{\bar{o}}])$$

since both $[s_o]$ and $[s_{\bar{o}}]$ have components in $\mathbb{Z}/4$ - grading $s_{min}(K)$

either sum or difference cancels in $Lee_1(K) \oplus Lee_{-1}(K)$

Algebraic Aside: Mapping cone

Let $f : A \longrightarrow B$ be a chain map between

$$A = \left(\bigoplus_i A^i, d_A\right)$$

$$B = \left(\bigoplus_i B^i, d_B\right)$$

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-1} \arrow[r, "d_A^{-1}"] \arrow[d, "f^{-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & A^0 \arrow[r, "d_A^0"] \arrow[d, "f^0"'] \arrow[dr, phantom, "\circlearrowleft"] & A^1 \arrow[r, "d_A^1"] \arrow[d, "f^1"'] \arrow[dr, phantom, "\circlearrowleft"] & A^2 \arrow[r] \arrow[d, "f^2"'] & \cdots \\
\cdots \arrow[r] & B^{-1} \arrow[r, "d_B^{-1}"'] & B^0 \arrow[r, "d_B^0"'] & B^1 \arrow[r, "d_B^1"'] & B^2 \arrow[r] & \cdots
\end{tikzcd}$$

commutes

The <u>mapping cone</u> of f is $C(f) := \left(\bigoplus_i (A^{i+1} \oplus B^i), \bigoplus_i \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix} \right)$

minus sign!

$$\begin{tikzcd}
\cdots \arrow[r] & A^{-1} \arrow[r, "-d_A^{-1}"] \arrow[d, "f"] & A^0 \arrow[r, "-d_A^0"] \arrow[d, "f"] & A^1 \arrow[r, "-d_A^1"] \arrow[d, "f"] & A^2 \arrow[r] \arrow[d, "f"] & \cdots \\
\cdots \arrow[r] & B^{-1} \arrow[r, "d_B^{-1}"] & B^0 \arrow[r, "d_B^0"] & B^1 \arrow[r, "d_B^1"] & B^2 \arrow[r] & \cdots
\end{tikzcd}$$

to make squares anti-commute

Exercise: $d^2_{C(f)} = 0$

In practice, we write the cone as

$$C(f) = \left(A \xrightarrow{f} B \right)$$

map becomes a part of the differential

Note: $\exists$ short exact sequence

$$0 \longrightarrow B \longrightarrow C(f) \longrightarrow A[-1] \longrightarrow 0$$

Example: 

$$V \otimes V \longmapsto V$$

$C(f)$ tells you to view the map as a differential

$V = \mathbb{Z}v_+ \oplus \mathbb{Z}v_-$

Khovanov chain cmplx associated to this diagram: $CKh($  $) = C\,(m : OO \longrightarrow$  $\{1\})$

Defn: A chain map $f : C \longrightarrow C'$ between filtered chain complexes is <u>filtered of degree k</u> if $f(F_i) \subseteq F'_{i+k}$

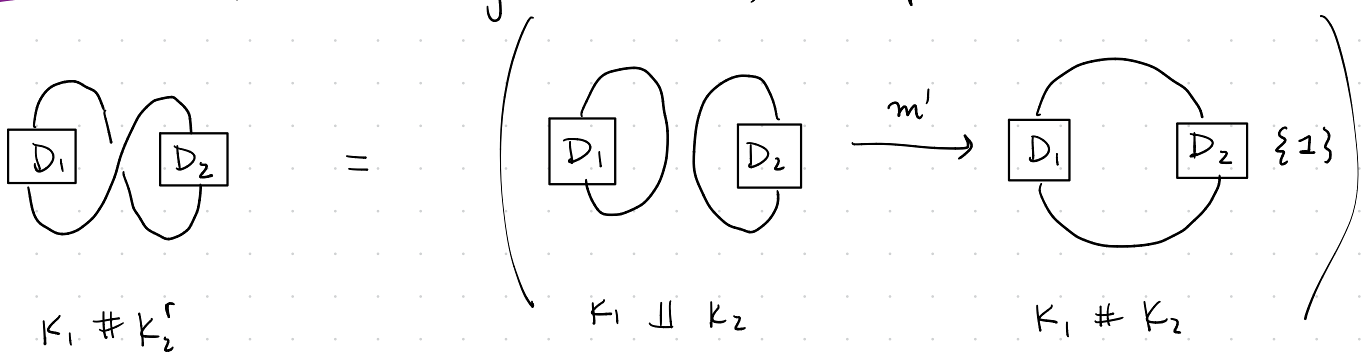
Lemma

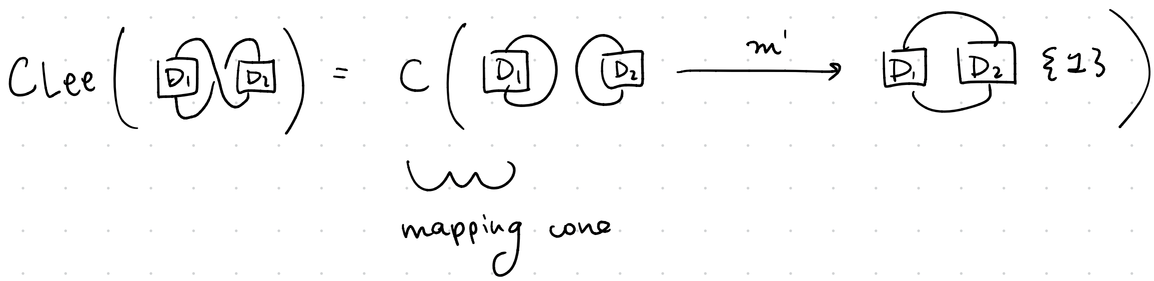
Let K_1, K_2 be knots. $\exists$ a short exact sequence

$$0 \longrightarrow \mathrm{Lee}(K_1 \# K_2) \xrightarrow{p^*} \mathrm{Lee}(K_1) \otimes \mathrm{Lee}(K_2) \xrightarrow{m^*} \mathrm{Lee}(K_1 \# K_2) \longrightarrow 0$$

where p^* and m^* have filtered degree -1

Proof: Let D_1, D_2 be diagrams for K_1, K_2 resp.





So we have a s.e.s.

$$0 \longrightarrow \mathrm{CLee}(K_1 \# K_2)\{1\} \xrightarrow{i} \mathrm{CLee}(K_1 \# K_2^r) \xrightarrow{p} \mathrm{CLee}(K_1 \amalg K_2) \longrightarrow 0$$

which induces a long exact sequence on homology

$$\begin{tikzcd}
\underset{\mathbb{Q}\oplus\mathbb{Q} \quad \dim = 2}{\mathrm{Lee}(K_1 \# K_2)} \arrow[rr, "i^*"] & & \underset{\mathbb{Q}\oplus\mathbb{Q} \quad \dim = 2}{\mathrm{Lee}(K_1 \# K_2^r)} \arrow[dl, "p^*"] \\
& \underset{\mathbb{Q}^4 \quad \dim = 4}{\mathrm{Lee}(K_1 \amalg K_2)} \arrow[ul] &
\end{tikzcd}$$

Exercise: $i^* = 0$ since $2+2=4$ (rank nullity, exactness ...)

and so l.e.s. of vector spaces splits, ie. we have a s.e.s.

$$0 \longrightarrow \mathrm{Lee}(K_1 \# K_2^r) \xrightarrow{p^*} \mathrm{Lee}(K_1 \amalg K_2) \xrightarrow{m^*} \mathrm{Lee}(K_1 \# K_2) \longrightarrow 0$$

What happens to the quantum grading?

Exercise: Both p^* and m^* are filtered of degree -1

Goal:

$\boxed{(*)\ \text{Proposition}}$ (K is a knot)

$$s_{max} = s_{min} + 2, \quad \text{or equivalently} \quad s = s_{max} - 1 = s_{min} + 1$$

Proof: Above lemma with $K_1 = K$
$K_2 = U$

$$0 \longrightarrow \mathrm{Lee}(K) \xrightarrow{p^*} \mathrm{Lee}(K) \otimes \mathrm{Lee}(U) \xrightarrow{m^*} \mathrm{Lee}(K) \longrightarrow 0$$

p^* and m^* are filtered of degree -1.

$\mathrm{Lee}(U) = \mathbb{Q}a \oplus \mathbb{Q}b$

to be continued!

Last time:

non-homogeneous basis $a = v_- + v_+$

$b = v_- - v_+$

to do this change of basis need to be over $\mathbb{Q}$ instead of $\mathbb{Z}$

Lee canonical generators s_o, $s_o \pm s_{\bar{o}}$ generate summands in

$$\mathrm{Lee}(K) \cong \mathrm{Lee}_1(K) \oplus \mathrm{Lee}_{-1}(K)$$

for knot, quantum grading is odd so the Lee homology splits like this in the $\mathbb{Z}/4$-graded theory

$$s_{min} = gr_q([s_o]) = gr_q([s_{\bar{o}}])$$

is the quantum grading of either s_o or $s_{\bar{o}}$

Goal:

$\boxed{(*)\ \text{Proposition}}$ (K is a knot)

$$s_{max} = s_{min} + 2, \quad \text{or equivalently} \quad s = s_{max} - 1 = s_{min} + 1$$

Proof of Prop: use Lemma with $K_1 = K$, $K_2 = U$

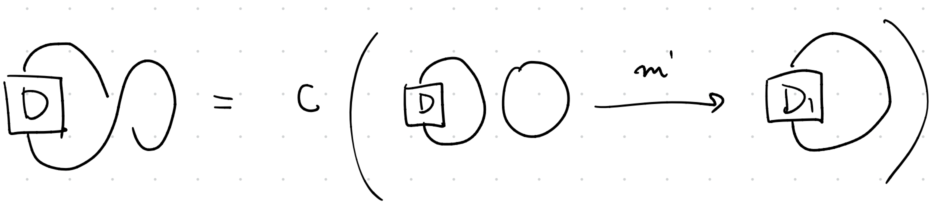
$$0 \longrightarrow \mathrm{Lee}(K) \xrightarrow{p^*} \mathrm{Lee}(K) \otimes \mathrm{Lee}(U) \xrightarrow{m^*} \mathrm{Lee}(K) \longrightarrow 0$$

p^* and m^* are filtered of degree -1.

We want to understand stuff in here!

$\mathrm{Lee}(U) = \mathbb{Q}a \oplus \mathbb{Q}b$

$\hookrightarrow$ crossingless diagram of K has orientations acc. to a and b and checkerboard coloring



One of $\{s_o, s_{\bar{o}}\}$ has a label a on the component where the connected sum appears. Call this generator s_a and the other s_b

$$gr_q([s_a + \varepsilon s_b]) = s_{max} \quad \text{for } \varepsilon = 1 \text{ or } -1$$

(we don't know which is gen. s_{max})

$$m^*([s_a + \varepsilon s_b] \otimes a) = [s_a] \quad (\text{by defn of } m)$$

Recall:

Exercise: m': $a \otimes a \mapsto 2a$

$a \otimes b,\ b \otimes a \mapsto 0$

$b \otimes b \mapsto -2b$

Since m^* is filtered of degree -1,

$$gr_q([s_a]) \geq gr_q([s_a + \varepsilon s_b] \otimes a) - 1$$

$$gr_q([s_a + \varepsilon s_b] \otimes a) \leq gr_q([s_a]) + 1$$

Recall: $s_{min}(K) = gr_q([s_o]) = gr_q([s_{\bar{o}}])$

Hence $s_{max}(K) - 1 \leq s_{min}(K) + 1$

Also recall: $s_{max}(K) \not\geq s_{min}(K)$

$$\Rightarrow s_{max}(K) = s_{min}(K) + 2$$

Exercise: Show that $s(RHT) = -\cancel{2} = 2$

↳ just find one of the generators $\{s_o, s_{\bar{o}}\}$

After discussion, this might be a positive 2

Properties of s

Exercise:

1. Suppose that K_+ and K_- differ by a single crossing change (positive to negative from K_+ to K_-)

   then $s(K_-) \leq s(K_+) \leq s(K_-) + 2$

2. $s(-K) = -s(K)$

3. $s(K_1 \# K_2) = s(K_1) + s(K_2)$

Theorem (Rasmussen)

$$\boxed{\left|\frac{s(K)}{2}\right| \leq g_4^{\text{Smooth}}(K)}$$

$\therefore \; s(4_1) = 0$

$s(U) = 0$

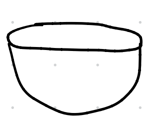
Sharp for torus knots

The proof relies on

- view minimal genus slice surface as a genus $g_4(K)$ cobordism between U and K

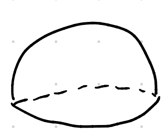
- Decompose cobordism into elementary cobordisms:

minima



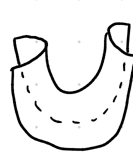
cup

maxima



cap

saddles — either merge or split 2 components



merge or split

Khovanov homology as (1+1)-TQFT

→ topological quantum field theory

hand wavy! (sorry!)

knots are 1-dim.

cobordisms sort of add a dimension (surface cobord. b/t knots)

Cobordism gives a map between Khovanov hom of the knots on ends which behaves nicely with product ⇒ id. on hom and "stacking" otherwise

Defn: a Frobenius system is the data $(R, A, \iota, m, \varepsilon, \Delta)$ consisting of

1. a commutative ground ring (e.g. $\mathbb{Z}, \mathbb{Q}$)

2. an R-algebra A, in particular

   a) the inclusion map $\iota : R \longrightarrow A$ that sends $\underbrace{1 \mapsto 1}_{\text{gives unit}}$

   b) multiplication map $m : A \otimes A \longrightarrow A$

3. co-multiplication map $\Delta : A \longrightarrow A \otimes A$ that is co-associative and co-commutative

i.e.

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] \arrow[dr, phantom, "\circlearrowright"] & A \otimes A \arrow[d, "id \otimes \Delta"] \\ A \otimes A \arrow[r, "\Delta \otimes id"'] & A \otimes A \otimes A \end{tikzcd}$$

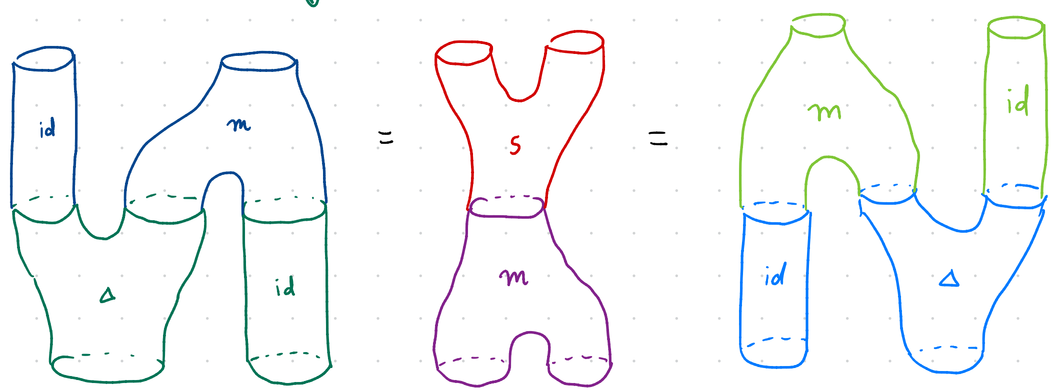
(Reversing all arrows would give associativity)

4. R-module $\underbrace{\text{co-unit}}_{\text{goes other way}}$ $\varepsilon : A \longrightarrow R$ $\quad (\varepsilon \otimes id) \circ \Delta = id$

A is a Frobenius algebra

i.e. it is both an algebra and a co-algebra and the following relation holds:

$$(\mathrm{id}_A \otimes m)\circ(\Delta \otimes \mathrm{id}_A) = \Delta \circ m = (m \otimes \mathrm{id}_A)\circ(\mathrm{id}_A \otimes \Delta)$$



Example: $(\mathbb{Z}, V, \iota, m, \varepsilon, \Delta)$ is a Frobenius system

$\iota: \mathbb{Z} \to V$, $1 \mapsto v_+$

$\varepsilon: V \to \mathbb{Z}$, $v_+ \mapsto 0$, $v_- \mapsto 1$

$m: v_+ \otimes v_+ \mapsto v_+$, $v_+ \otimes v_-, v_- \otimes v_+ \mapsto v_-$, $v_- \otimes v_- \mapsto 0$

Example: $(\mathbb{Q}, V, \iota, m', \varepsilon, \Delta')$ is a Frobenius system

Key idea: A cobordism $F: K_0 \to K_1$ induces chain maps

$$CKh(F): CKh(K_0) \longrightarrow CKh(K_1)$$
$$CLee(F): CLee(K_0) \longrightarrow CLee(K_1)$$

$CLee(F)$ is a filtered map of degree $\chi(F)$

(see exercise 3.3.11 in Zhang)

Rasmussen's proof that $\left|\frac{s(K)}{2}\right| \leq g_4(K)$ relies on showing that map induced by cobordism is non-zero.

Corollary

$\frac{s}{2} : \mathcal{C} \longrightarrow \mathbb{Z}$ is a surjective homomorphism

↳ well-defined b/c it's zero on slice knots

s is additive under connected sum

$s(RHT) = -2$ (gen. of the integers)

Theorem

If D is a positive diagram for a positive knot K, then

knot who has a projection with all + crossings

$$s(K) = gr_q(s_o) + 1$$

Proof: The oriented resolution D_o is the unique resolution at the left most vertex of the cube of resolutions.

Then there are no differentials into this homological grading, so s_o and $s_{\bar{o}}$ are alone in their respective homology classes.

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Exercise: Show that $s(T_{p,q}) = (p-1)(q-1)$

and conclude that

$$\underbrace{u(T_{p,q})}_{\text{by exercise below!}} = g_4^{smooth}(T_{p,q}) = \underbrace{\frac{(p-1)(q-1)}{2}}_{\text{this resolves the Milnor Conjecture}}$$

this is the same as the Seifert genus for torus knots

Exercise: $g_4^{smooth}(K) \leq u(K)$

unknotting number $u(K)$ is the min. no. of crossing changes to get to unknot

Exercise: (Maybe hard?)

If K is alternating, then $s(K) = -\sigma(K)$

Defn: A slice-torus invariant is a concordance homomorphism

$$\phi : \mathcal{C} \longrightarrow \mathbb{R}$$

satisfying

1. (slice) $\phi(K) \leq g_4(K) \quad \forall K$

2. (torus) $\phi(T_{p,q}) = g_4(T_{p,q}) \quad \forall p,q > 0$ coprime

Example: $\frac{s}{2}$ is a slice-torus invariant

s has a lot in common with another concordance invariant τ defined by Ozsváth-Szabó coming from Heegaard Floer homology. In particular, τ is also a slice-torus invariant.

(there are $\mathrm{Wh}(K)$ where s and τ differs)

s and τ are not related by any formula:

s and τ are linearly independent in the sense that

$$\frac{s}{2} \oplus \tau : \mathcal{C} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \quad \text{surjective}$$

Ribbon Concordances and Khovanov Homology

No maxima, so order matters!

Theorem: (Levine-Zemke)

If $C : K_0 \longrightarrow K_1$ is a ribbon-concordance, then

$$Kh(C) : Kh(K_0) \longrightarrow Kh(K_1)$$

is injective with left inverse $Kh(\bar{C})$

C 

$\bar{C}$ 

$\bar{C}$ upside down (w/ oppossite orientation)