8803

Week 7

Monday pg 2

Wednesday pg 10

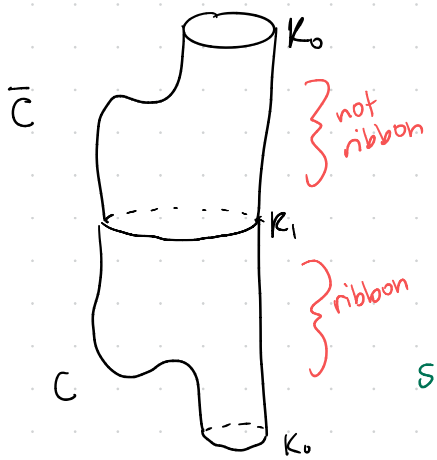
No Office Hours today

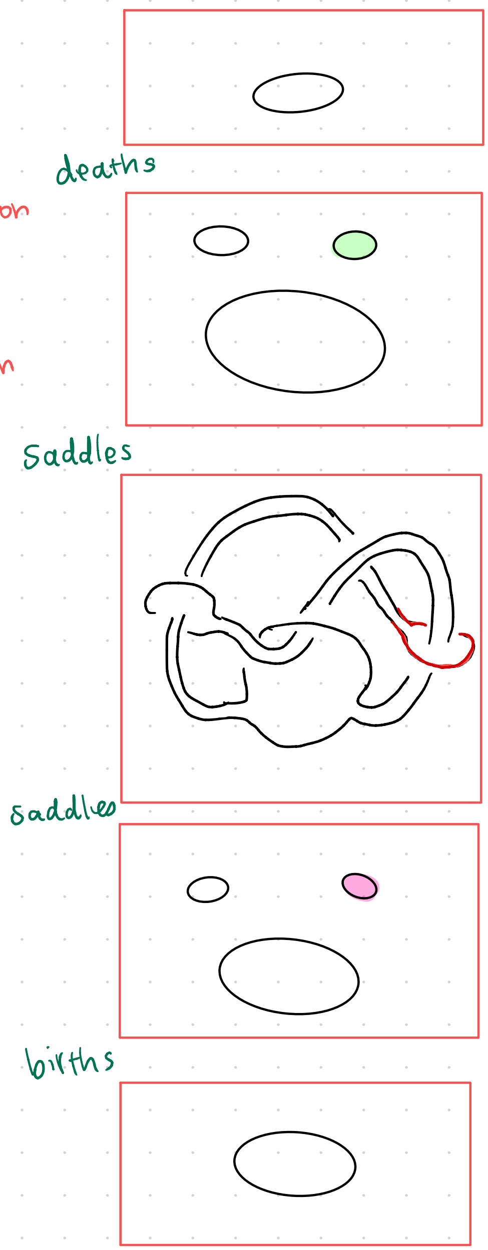
office Hours on Tuesday 2-3 pm and Wednesday 10 45 - 11 45 am

Last time: ribbon concordances and Khovanov homology

Theorem: (Levine-Zemke)

If $C: K_0 \to K_1$ is a ribbon concordance, then $Kh(C): Kh(K_0) \longrightarrow Kh(K_2)$ is injective with left inverse $Kh(\overline{C})$

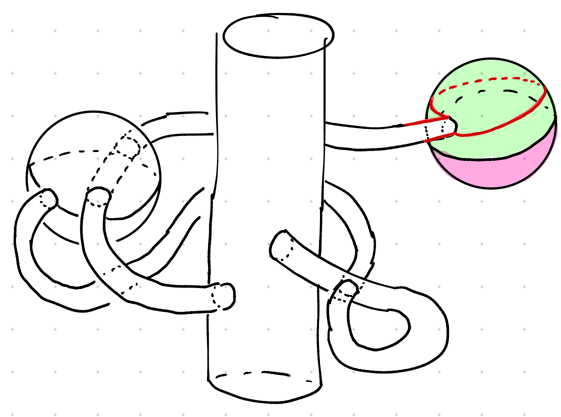




Rmk: when you compose something nice happens

Rmk: each birth is paired to a death

Idea:



time frames thinking of cobordance as a movie

Idea: birth-deaths determine 2-spheres that are tubed on to $K_0 \times I$ by tubes formed by saddles paired with their duals

Key Lemma (Zemke)

Let $C: K_0 \to K_1$ be a ribbon concordance with n births, n saddles

$$\bar{C} : K_1 \longrightarrow K_0$$ (upside down and opposite orientation)

Then $\bar{C} \circ C : K_0 \longrightarrow K_0$ is isotopic to $K_0 \times I$ with n unknotted, (geometrically) unlinked 2-spheres tubed on

Idea of proof of Theorem:

$$Kh(\bar{C}) \circ Kh(C) = Kh(\bar{C} \circ C)$$

onto injective

behavior of Khovanov homology under neck-cutting

$$= Kh\left(id_{K_0} \sqcup n \text{ unknotted unlinked } S^2\text{'s} \right)$$

another property of Kh. hom

$$= Kh(K_0 \times I)$$

$$= id_{Kh(K_0)}$$

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Upshot: ribbon concordances induce particularly nice maps on Khovanov homology

Q: What about ribbon concordances and classical knot invariants?

things such as Seifert forms and earlier.
(pre-Heegaard Floer, pre-Khovanov)

Proposition (Gordon)

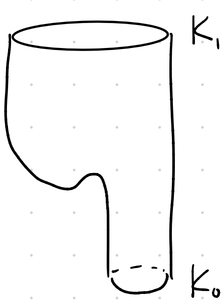
$X = S^3 \times I - C$

$Y_i = S^3 - K_i$

If $C : K_0 \to K_1$ is a ribbon concordance, then

1. $\pi_1(Y_0) \hookrightarrow \pi_1(X)$

2. $\pi_1(Y_1) \twoheadrightarrow \pi_1(X)$



$Y_1 = S^3 - K_1$

$X = S^3 \times I - \nu(C)$

$Y_0 = S^3 - K_0$

Proof of 2.

4-dim k-handle added to C

4-dim $(k+1)$-handle added to X

$X = (Y_0 \times I) \cup$ 1-handles $\cup$ 2-handles

dually: $X = (Y_1 \times I) \cup$ 2-handles $\cup$ 3-handles

$\Rightarrow \pi_1(Y_1) \longrightarrow \pi_1(X)$ surjective.

1. Exercise:

1. $H_*(Y_0) \longrightarrow H_*(X)$ isomorphism

2. In $X = (Y_0 \times I) \cup$ 1-handles $\cup$ 2-handles the no. of 1-handles = no. of 2-handles and the 2-handles must cancel 1-handles homologically

Hence, $\pi_1(X) = \langle \pi_1(Y_0) * F \rangle / \langle r_1, \dots, r_n \rangle$

$r_j \in \pi_1(Y_0) * F$

$\varepsilon_i(r_j) =$ exponent of x_i in r_j

$n \times n$ matrix $(\varepsilon_i(r_j))$ has determinant ± 1

WTS $\pi_1(Y_0) \longrightarrow \pi_1(X)$ injective

about homomorphisms to a group

Defn: A group G is residually finite if $\forall g \in G, g \neq 1$ $\exists$ homomorphism $h: G \longrightarrow$ finite group s.t. $h(g) \neq 1$.

Proposition (Thurston)

$\pi_1(Y_0)$ is residually finite

Suppose $z \in \ker\left(\pi_1(Y_0) \longrightarrow \pi_1(X)\right)$

If $z \neq 1$, then since $\pi_1(Y_0)$ is residually finite, $\exists$

$$\rho : \pi_1(Y_0) \longrightarrow G, \quad G \text{ finite such that } \rho(z) \neq 1.$$

Then we have $\pi_1(X) \longrightarrow \overbrace{(G * F)/\langle \rho'(r_1), \dots, \rho'(r_n) \rangle}^{H}$

where $\rho' : \pi_1(Y_0) * F \longrightarrow G * F$ is induced by ρ

Group theory result of Gerstenhaber-Rothaus implies that

$G \to H$ is injective (uses fact that $(\varepsilon_i(r_j'))$ has determinant ± 1)

$$\begin{tikzcd} z \arrow[d, mapsto] & \pi_1(Y_0) \arrow[r, "z \mapsto 1"] \arrow[d, "\rho"'] & \pi_1(X) \arrow[d] \\ \neq 1 & G \arrow[r, hook] & H \end{tikzcd}$$

$\otimes$ contradiction. Hence $\pi_1(Y_0)$ is injective.

Defn: A homotopy ribbon concordance from K_0 to K_1 is a locally flat concordance C from K_0 to K_1 s.t.

1. $\pi_1(Y_0) \hookrightarrow \pi_1(X)$
2. $\pi_1(Y_1) \twoheadrightarrow \pi_1(X)$ where $Y_i = S^3 - K_i$

$$X = S^3 \times I - C$$

We will write $K_0 \underset{top}{\leq} K_1$ if $\exists$ a homotopy ribbon concordance from K_0 to K_1

Write $K_0 \leq K_1$ if $\exists$ a ribbon concordance from K_0 to K_1

<u>Observe</u>: $\underset{\text{(smooth)}}{K_0 \leq K_1} \implies K_0 \underset{top}{\leq} K_1$

Theorem (Agol 2022)

Ribbon concordance is a partial order

i.e. $K_0 \leq K_1$ and $K_1 \leq K_0 \implies K_0 = K_1$

Resolves a conjecture of Cameron Gordon

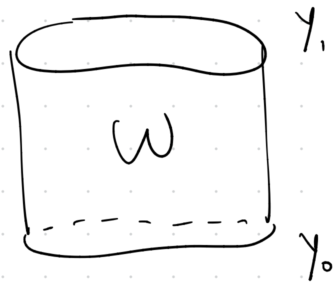
<u>proof</u>: relies on representation varieties of knot group to $SO(N)$

Homology Cobordism Group

Closed, connected, oriented 3-manifolds

(compact, without boundary)

Defn: Two 3-mfds Y_0 and Y_1 are cobordant if $\exists$ smooth compact 4-mfd W s.t. $\partial W = -Y_0 \sqcup Y_1$



Remark: This is an equivalence relation

Proposition

Every 3-mfd bounds a smooth compact 4-mfd

Proof: By Lickorish-Wallace, every 3-mfd Y is integral surgery on a link L in S^3

Let X be a 4-mfd obtained by attaching framed 2-handles along $L \in \partial B^4$. Then $\partial X = Y$.

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$\boxed{\text{Corollary}}$

Any two 3-mfd are cobordant

Defn: Two 3-mfds Y_0 and Y_1 are $\mathbb{Z}$-homology cobordant if $\exists$ smooth compact 4-mfd W s.t.

1. $\partial W = -Y_0 \sqcup Y_1$

2. $i_*: H_*(Y_i;\mathbb{Z}) \longrightarrow H_*(W;\mathbb{Z})$ is an isomorphism

"W looks like a product in terms of its homology"

Remarks:

1. Can replace $\mathbb{Z}$ with $\mathbb{Q}$, $\mathbb{Z}_p$, or any ring R

2. Homology cobordism is an equivalence relation.

Example: Y 3mfd

$Y \times I$ is a homology cobordism

Last time:

Y_0, Y_1 $\mathbb{Z}$-homology cobordant if $\exists$ smooth, compact W s.t.

1. $\partial W = -Y_0 \amalg Y_1$

2. $i_*: H_*(Y_i; \mathbb{Z}) \longrightarrow H_*(W; \mathbb{Z})$ isomorphism $\quad i=0,1$

"W looks like a product in terms of its homology"

Remark: can replace $\mathbb{Z}$ with another ring (eg. $\mathbb{Q}$, $\mathbb{Z}/_p\mathbb{Z}$)

Example:

Y any 3-mfd

$Y \times I$ is an R-homology cobordism for any R

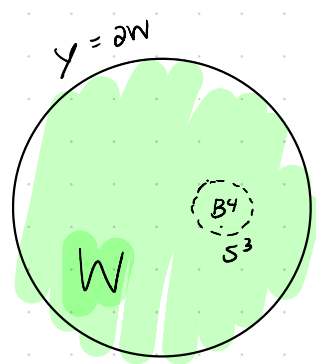
Example:

4-mfd w/ same $\mathbb{Z}$ homology as B^4

If Y bounds a $\mathbb{Z}H_*B^4$

$\Longleftrightarrow$

$Y \underset{\substack{\mathbb{Z}H_* \\ \text{cobord}}}{\sim} S^3$



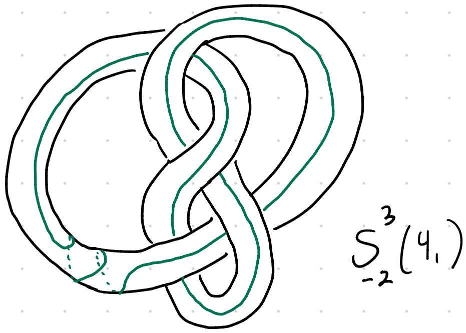
remove B^4

(works for any R)

Exercise:

If $K_0 \underset{\text{smooth}}{\sim} K_1$, then $S^3_{p/q}(K) \underset{\substack{\mathbb{Z}H_* \\ \text{cob}}}{\sim} S^3_{p/q}(K_1)$

Recall: $S^3_{p/q}(K) = S^3 - \nu(K) \cup_\varphi S^1 \times D^2$



$p\mu + q\lambda$ bounds a disk in $S^1 \times D^2$

Note:

(q longitudes are already null-homologous)

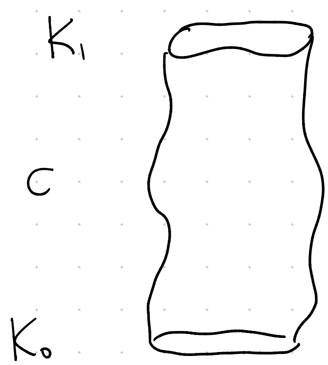
↳ 0 framed longitude bounds a Seifert sfc.

1. $H_1(S^3_{p/q}(K); \mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$

2. If $p = \pm 1$ then $S^3_{p/q}(K)$ is an $\mathbb{Z}HS^3$

3. If $p \neq 0$ then $S^3_{p/q}(K)$ is a $\mathbb{Q}HS^3$

To see that these 3-mfds are homology cobordant, surger the concordance to build W



Exercise:

If K is smoothly slice, then $\Sigma_q(K)$ $q = p^n$ for prime p bounds a $\mathbb{Q}HB^4$. Here $\Sigma_q(K)$ denotes the q-fold cyclic branch cover of K

Note:

$\Sigma_q(K) = \big(q\text{-fold cyclic cover of } S^3 - \nu(K)\big) \cup \big(S^1 \times D^2\big)$

core of

$\Sigma_q(K)$ pre-image K is $S^1 \times \{0\}$

$\Sigma_q(K) \xrightarrow{\pi} S^3$

π is modeled on the map $z \longmapsto z^q$ (in $\mathbb{C}$)

along pt $\times D^2$

Proof idea: take q-fold cyclic cover of B^4 branched over slice disk

Need $q = p^n$ to guarantee that $\Sigma_q(K)$ is $\mathbb{Q}HS^3$

Non-example: $\Sigma_6(T_{2,3})$ is not a $\mathbb{Q}HS^3$

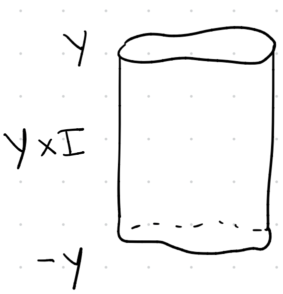
$H_1 \approx \mathbb{Z}$

(see Rolfsen 10.D)

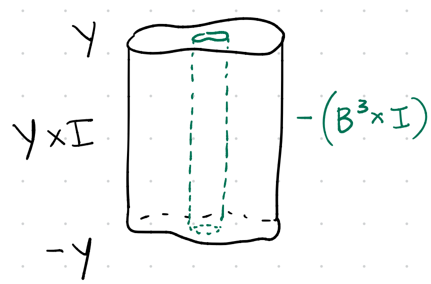
Exercise:

Let Y be a RHS^3, then $Y \# -Y$ bounds a RHB^4

Idea:



then drill out a tube



$W = (Y - B^3) \times I$

$\partial W = Y \# -Y$

Note: Y_1, Y_2 RHS$^3 \Longrightarrow Y_1 \# Y_2$ a RHS^3

For now, lets focus on ring $\mathbb{Z}$

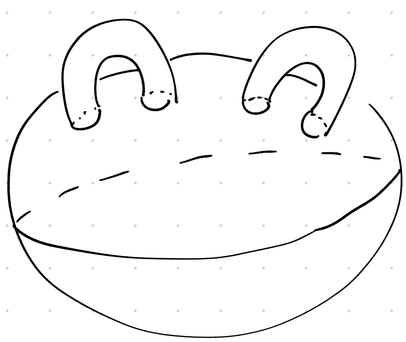
Consider $(\{\mathbb{Z}HS^3\text{'s}\}, \#)$

Q: Y $\mathbb{Z}HS^3$, $Y \neq S^3$ does there exist Y' s.t. $Y \# Y' \overset{!}{=} S^3$?

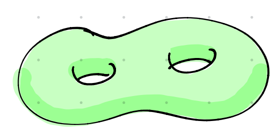
A: No

Recall, A Heegaard splitting of a 3-mfd Y is a decomposition $Y = H_1 \cup_\varphi H_2$ where H_i is a handlebody of genus g and $\varphi : \partial H_1 \longrightarrow \partial H_2$ is an orientation reversing homeomorphism

Handlebody:



$\cong$



handlebody of genus 2

Every 3-mfd has a Heegaard splitting. Consider triangulation of 3-mfd, then look at 1-skeleton

Ex: $S^3 = B^3 \cup B^3$

Ex: $S^1 \times S^2 = (S^1 \times D^2) \cup_\varphi (S^1 \times D^2) \qquad \varphi = -id_{S^1 \times D^2}$

Exercise: Find φ s.t.

$$L(p,q) = (S^1 \times D^2) \cup_\varphi (S^1 \times D^2)$$

Example:

$$S^3 = (S^1 \times D^2) \cup_\varphi (S^1 \times D^2)$$

$$\lambda \longmapsto \mu$$

$$\mu \longmapsto \lambda$$

Defn: The Heegaard genus of 3-mfd Y is the minimum Heegaard genus over all Heegaard splittings

Example: S^3 is the only 3-mfd w/ Heegaard genus zero

Example: Heegaard genus of $S^1 \times S^2$, $L(p,q)$ is 1

Theorem (Haken)

$\boxed{\text{Heegaard genus is additive under connected sum}}$

Proof Idea: the connect sum S^2 can be isotoped to intersect Heegaard surface in a single circle.

So Heegaard splitting for $Y_1 \# Y_2$ restricts to a Heegaard splitting for Y_1 and for Y_2

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Hence, using Heegaard genus, we see that if $Y \neq S^3$, then $\nexists\ Y'$ s.t. $Y \# Y' = S^3$

However for Y a $\mathbb{Z}HS^3$, $Y \# -Y \underset{\substack{\mathbb{Z}H_* \\ \text{cob}}}{\sim} S^3$ since $Y \# -Y$ bounds $\mathbb{Z}HB^4$

Defn: The $\mathbb{Z}$-homology cobordism group is

$$\Theta^3_{\mathbb{Z}} = \left(\{\mathbb{Z}HS^3\text{'s}\} \Big/ \underset{\substack{\mathbb{Z}H_* \\ \text{cob}}}{\sim}\ ,\ \# \right)$$

id $[S^3]$

inverse of $[Y]$ is $[-Y]$

Q: Is $\Theta^3_{\mathbb{Z}}$ nontrivial?

A: Yes

Rokhlin invariant

Y $\mathbb{Z}HS^3$ (or more generally a spin 3-mfd)

$$\mu(Y) = \frac{\sigma(X)}{8} \quad \text{where } X \text{ is a spin 4-mfd w/ } \partial X = Y$$

Remarks:

second Stiefel Whitney class in $H^2(X;\mathbb{Z})$

1. X spin $\Longleftrightarrow$ $w_2(X) = 0$

2. If X simply connected, then $w_2(X) = 0 \Longleftrightarrow$ intersection form of X is even

3. a) $\sigma(X)$ is divisible by 8

   b) $\sigma(X)$ mod 16 depends only on Y (and not X)

$$\Rightarrow \mu(Y) = \frac{\sigma(X)}{8} \in \mathbb{Z}/2$$

4. $\mu(Y_1 \# Y_2) = \mu(Y_1) + \mu(Y_2)$

Exercise:

Check that $\mu : \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}/2$ is a well-defined surjective homomorphism

see Remark #4

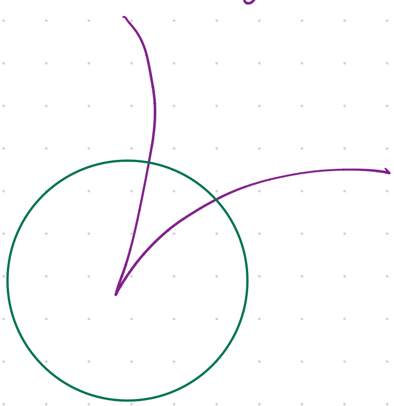
$\mu(\Sigma(2,3,5)) = 1$

Recall:

small sphere at origin

$$\Sigma(p,q,r) = \{x^p + y^q + z^r = 0\} \cap S^5_\varepsilon \subset \mathbb{C}^3$$

zero set of a polyn.



$\Sigma(p,q,r)$ is a **Brieskorn homology sphere** for p,q,r relatively prime

[Fintushel-Stern 1985] showed $\Theta^3_{\mathbb{Z}}$ infinite

[Furuta, Fintushel-Stern 1990] $\Theta^3_{\mathbb{Z}}$ infinitely generated

[Frøyshov 2002] $\Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}$ surjective homomorphism

all proved with gauge theory

[Dai-Hom-Stoffreger-Truong] $\Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}^\infty$ surj. homom.

uses involutive Heegaard Floer hom.

$\Sigma(2i+1, 4i+1, 4i+3)_{i>0}$ generate the DHST infinite rank summand

Open Question: $\exists$ nontrivial elements of finite order in $\Theta^3_{\mathbb{Z}}$?

Note: $Y \cong -Y \implies [Y \# Y] = [S^3]$ in $\Theta^3_{\mathbb{Z}}$ (many candidates!)

However, it is very difficult to show that Y is nontrivial in $\Theta^3_{\mathbb{Z}}$

Theorem (Manolescu 2013)

If $\mu(Y)=1$, then Y is not order 2 in $\Theta^3_{\mathbb{Z}}$

(i.e. $[Y \# Y] \neq [S^3]$ in $\Theta^3_{\mathbb{Z}}$)

Main Idea:

define a homology cobordism invariant $\beta \in \mathbb{Z}$ s.t.

1. $\beta(-Y) = -\beta(Y)$
2. $\beta(Y) \bmod 2 = \mu(Y)$
3. If $Y_0 \underset{\mathbb{Z}H_x \text{ cob}}{\sim} Y_1$ then $\beta(Y_0) = \beta(Y_1)$

$$\mu(Y) = 1 \overset{2}{\implies} \beta(Y) \text{ odd}$$

$$\implies \beta(-Y) \overset{1}{=} -\beta(Y) \neq \beta(Y)$$

3 $\implies Y \underset{\substack{\mathbb{Z}H_* \\ \text{cob}}}{\not\sim} -Y$ i.e. Y is not order 2 in $\Theta^3_{\mathbb{Z}}$ ///

By earlier work of Galewski-Stern and Matumoto, Manolescu's result implies:

Theorem

There exist non-triangulable n-dimensional topological manifolds $\forall \; n \geq 5$

Triangulations

(Manolescu lectures on the Triangulation Conjecture)

Defn: A **simplicial complex** $K = (V, S)$ consists of

- V = finite collection of vertices
- S = finite collection of simplices (where a simplex is an element of $\mathcal{P}(V)$) power set

such that $\sigma \in S$ and $\tau \subset \sigma$, then $\tau \in S$

We call (V,S) an abstract simplicial complex

To (V,S) we can associate its geometric realization K constructed inductively on $d \geq 0$ by attaching a d-dim simplex for each $\sigma \in S$ of cardinality d

Example: $K = (V,S)$

$V = \{1,2,3,4\}$

$S = \{ \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\} \}$

geom. realization: