8803

Week 8

Last time:

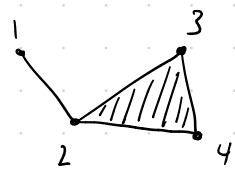
[Manolescu] $\mu(Y)=1 \implies Y$ not order 2 in $\Theta^3_{\mathbb{Z}}$

By work of Galewski–Stern, Matumoto, this implies $\exists$ non-triangulable n-dim top. manifolds $\forall\ n \geq 5$

Simplicial complex $K=(V,S)$

Example:

$V=\{1,2,3,4\}$



$S=\{\{1\},\{2\},\{3\},\{4\},$
$\{2,3\},\{2,4\},\{3,4\},$
$\{2,3,4\}\}$

S prime

The closure of a subset $S'\subset S$ is

$$Cl(S')=\{\tau\in S \mid \tau\subseteq\sigma\in S'\}$$

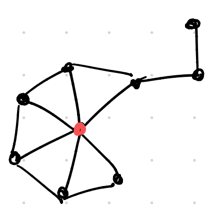
simplex

The star of a simplex $\tau\in S$ is

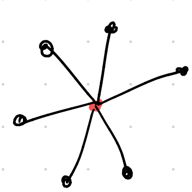
$$St(\tau)=\{\sigma\in S \mid \tau\subseteq\sigma\}$$

Example:

$\tau=$

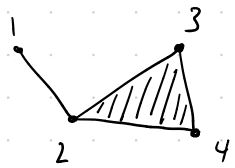


$St(\tau)=$



Example:

$V = \{1, 2, 3, 4\}$



$$S = \{ \{1\}, \{2\}, \{3\}, \{4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{2,3,4\} \}$$

$$St(3) = \{ \{3\}, \{2,3\}, \{3,4\}, \{2,3,4\} \}$$

$$Cl(St(3)) = \{ \{2\}, \{3\}, \{4\}, \{2,3\}, \{3,4\}, \{2,4\}, \{2,3,4\} \}$$

The *link* of a simplex $\tau \in S$ is

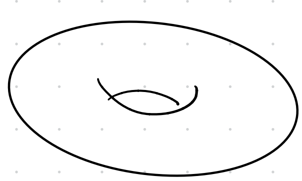
$$Lk(\tau) = \{ \sigma \in Cl(St(\tau)) \mid \tau \cap \sigma = \phi \}$$

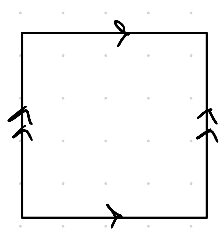
$$Lk(\{3\}) = \{ \{2\}, \{4\}, \{2,4\} \}$$

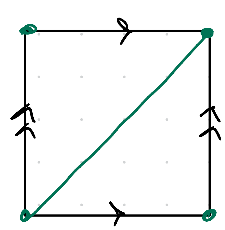
$$Lk(\{2\}) = \{ \{1\}, \{3\}, \{4\}, \{3,4\} \}$$

Defn: A *triangulation* of a topological space X is a homeomorphism from X to a simplicial complex

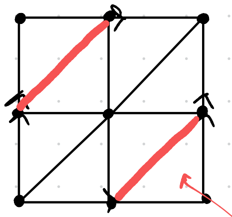
Ex:





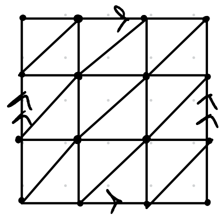


is not a simplicial complex because any pair of vertices does not uniquely define an edge.

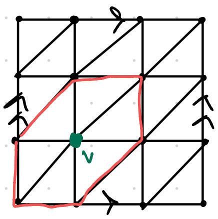


Also not a triangulation

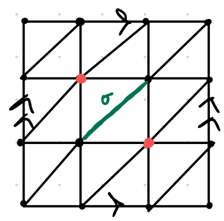
consider edge



Yes, a triangulation.



Lk(v)



Lk(σ)

Exercise:

1. If K is a triangulation of a topological manifold M^n and $\sigma \in K^{n-k}$, then $Lk(\sigma)$ is $\cong H_* S^{k-1}$

2. A triangulation on M induces a triangulation on its $\underbrace{\text{suspension } \Sigma M}$ and the link of a cone point is M

$M \times I$ with

$M \times \{0\}$, $M \times \{1\}$ collapsed to a point called a cone point

Some categories of manifolds:

- topological manifolds: transition functions on charts are continuous
- PL manifolds: transition fn's are piecewise linear
- Smooth manifolds: transition fn's are C^∞ smooth

Defn: A triangulation is combinatorial if the link of every simplex (or the link of every vertex) is PL-homeomorphic to a sphere

Observe: If a space X admits a combinatorial triangulation then X is a PL-manifold

↳ Converse is also true.

Example:

Non PL triangulation of a topological manifold

P = homology sphere with nontrivial π_1

eg. Poincaré homology sphere

suspension

Fact 1: ΣP is not a manifold (except when P is sphere)

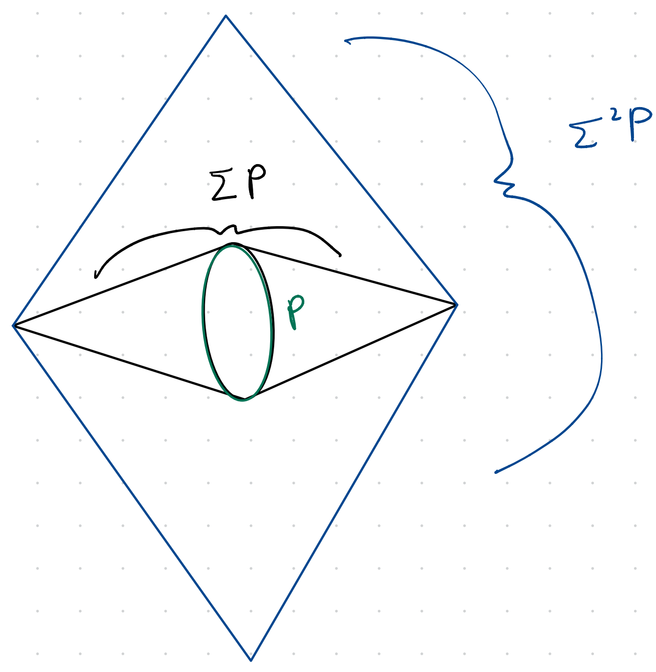
Fact 2: (Double suspension theorem) (Edwards 1980, Cannon 1979)

$\Sigma^2 P$ is a topological manifold homeomorphic to a sphere

Take a triangulation of P

This induces a triangulation on $\Sigma^2 P$

but this triangulation is not combinatorial : link of cone point is ΣP which is not even a manifold hence not PL homeo to a sphere.



Question : (Poincaré 1899)

Does every smooth manifold admit a triangulation

A : [Cairns 1935, Whitehead 1946] Yes

Every smooth manifold has a PL structure and hence is triangulable

Question: (Kneser 1926)

Does every topological manifold admit a triangulation?

A: Depends on dimension:

$n = 0, 1$ yes, trivial

$n = 2$ [Rado 1925] Yes, every surface has a PL-structure

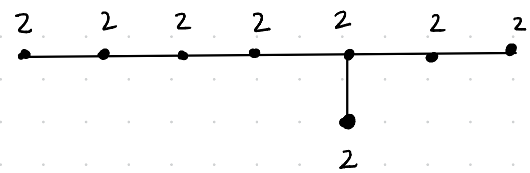
$n = 3$ [Moise 1952] Yes, every 3-manifold has a smooth structure

$n = 4$ [Casson] No, using Casson invariant, you can show Freedman's E_8 is not triangulable

(Rokhlin invariant shows that Freedman's E_8 manifold has no smooth structure)

Freedman's E_8:

- plumbing diagram =



- boundary is Poincaré $H_* S^3$

[Freedman] every $\mathbb{Z}HS^3$ bounds a compact, contractible topological 4-mfd

$n \geq 5$ [Manolescu 2013] No.

Question:

Does every topological manifold admit a PL structure?

A: depends on dimension

$n = 0, 1, 2, 3$ Yes, as above

$n = 4$ No, E_8 has no PL-structure

$n \geq 5$ [Kirby-Siebenmann] No

M topological manifold

Kirby-Siebenmann invariant $\Delta(M) \in H^4(M; \mathbb{Z}/2)$

$n \geq 5 \qquad \Delta(M) = 0 \iff M$ admits a PL-structure

$n = 4 \qquad \Delta(M) = 0 \Leftarrow M$ admits a PL-structure

Example: $\Delta(S^1 \times E_8) \neq 0$

so $S^1 \times E_8$ is a topological manifold with no PL-structure

More generally, $\Delta(T^{n-4} \times E_8) \neq 0$ for $n \geq 5$ is an n-dim mfd with no PL-structure

Kirby-Siebelmann invariant

M^n top. manifold $n \geq 5$

diagonal $D \subset M \times M$

$\nu(D)$ is an $\mathbb{R}^n$-bundle over M topological tangent bundle of M

$TOP(n)$ = homeomorphisms of $\mathbb{R}^n$ fixing 0

$TOP = \lim_{n \to \infty} TOP(n)$ infinite dim. top. group

$BTOP$ = classifying space of TOP

i.e. $\begin{array}{c} TOP \\ \downarrow \\ BTOP \end{array}$ weakly contractible space on which TOP acts properly and freely and $\exists\ \Phi : M \longrightarrow BTOP$ s.t. TM is the pullback:

$$\begin{tikzcd} TM \arrow[r] \arrow[d] & ETOP \arrow[d] \\ M \arrow[r, "\Phi"] & BTOP \end{tikzcd}$$

any bundle will be a pullback

$PL(n) =$ PL-homeomorphisms of $\mathbb{R}^n$ fixing 0

$PL = \lim_{n \to \infty} PL(n) \subset TOP$

Fibration:

$$\begin{tikzcd} TOP/PL \arrow[r] & BPL \arrow[d] \\ & BTOP \end{tikzcd}$$

$K(\mathbb{Z}/2;3) =$

$$\begin{tikzcd} & ETOP \arrow[dl] \arrow[dr] & \\ BPL \arrow[rr] & & BTOP \end{tikzcd}$$

(see more in Manolescu's lecture notes)

Obstruction theory: discusses when you can build a lift

$$\begin{tikzcd} & BPL \arrow[d] \\ M \arrow[ur, dashed] \arrow[r, "\Phi"] & BTOP \end{tikzcd}$$

How would you try to lift this map?

Possibly want to induct along n-skeleton

to an n simplex, lift to fiber

$\underbrace{n+1 \text{ simplex boundary}}_{\text{boundary is an } n\text{-sphere}}$ lifts to $\pi_n(\text{fiber})$

something sending simplex to a group sounds like a cochain.

(magic: it's actually a cocycle)

$\sigma^{n+1} \longmapsto \pi_n(F)$

$H^{n+1}(M; \pi_n(TOP/PL))$

but recall fiber is a $K(\mathbb{Z}/2;3)$

coming from TOP/PL being a $K(\mathbb{Z}/2;3)$

$$\Delta(M) \in H^4(M;\mathbb{Z}/2) = H^4(M;\pi_3(TOP/PL))$$

obstruction to lifting Φ

More concrete description of $\Delta(M)$, when M^n has a triangulation (not necessarily PL), $n \geq 5$

for simplicity, assume M orientable.

$$c(K) = \sum_{\sigma \in K^{n-4}} [Lk(\sigma)]_\sigma \in H_{n-4}(M;\theta^3_{\mathbb{Z}}) \underset{P.D.}{\cong} H^4(M;\theta^3_{\mathbb{Z}})$$

short exact sequence:

Rokhlin invariant

$$0 \longrightarrow \ker\mu \longrightarrow \theta^3_{\mathbb{Z}} \xrightarrow{\mu} \mathbb{Z}/2 \longrightarrow 0$$

induces a long exact sequence on cohomology:

$$\cdots \longrightarrow H^4(M;\theta^3_{\mathbb{Z}}) \xrightarrow{\mu} H^4(M;\mathbb{Z}/2) \xrightarrow{\delta} H^5(M;\ker\mu) \longrightarrow \cdots$$

$$\begin{array}{ccc} H^4(M;\theta^3_{\mathbb{Z}}) & & \\ \cup\!\!\shortmid & & \\ c(K) & \longmapsto & \Delta(M) \end{array}$$

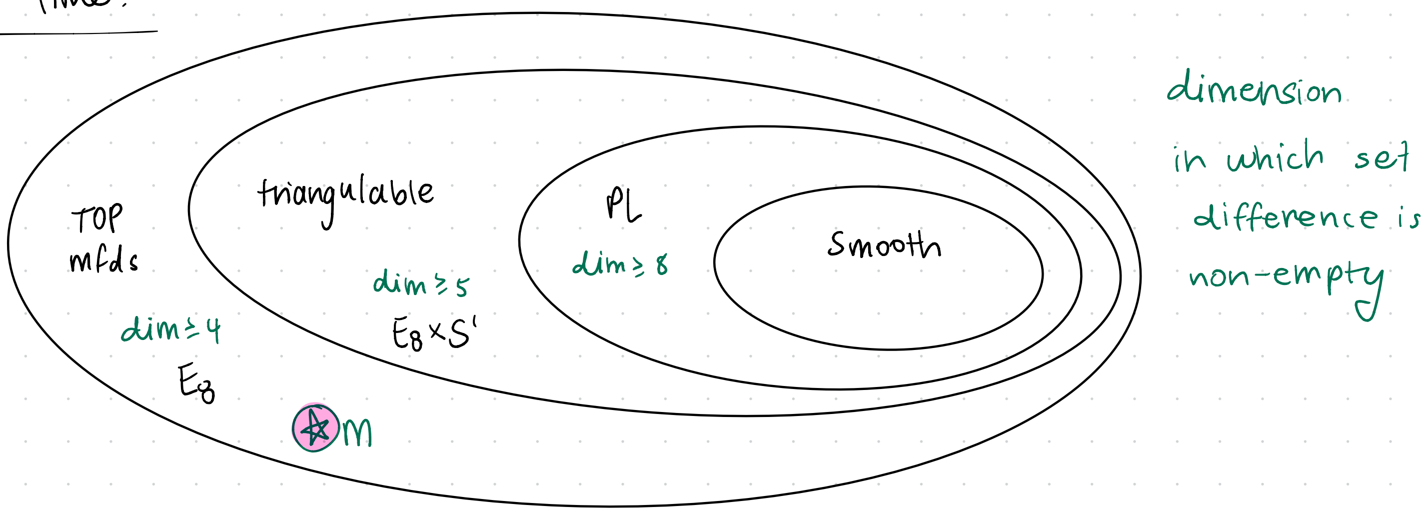
i.e. $\mu(c(K)) = \Delta(M)$

Observe:

K combinatorial $\Longrightarrow c(K) = 0$

$\mu(c(K)) = \Delta(M) = 0 \iff M$ admits a combinatorial triangulation (possibly different from K)

Last time:



Kirby-Siebelmann invariant

M^n top manifold $n \geq 5$

$\Delta(M) \in H^4(M; \mathbb{Z}/2)$

$n \geq 5 \qquad \Delta(M) = 0 \iff M$ admits PL structure

$n = 4 \qquad \Delta(M) = 0 \Longleftarrow M$ admits PL structure

So in particular, $\Sigma^2 P$ admits a PL structure)

Idea that $TOP/PL = K(\mathbb{Z}/2, 3)$ Rudyak's survey article (PL-structures on Top mfds)

Rough sketch:

$$\varphi : \pi_n(TOP/PL) \longrightarrow S_{PL}(T^k \times D^n) \text{ injective}$$

$S_{PL}(M)$ = set of homotopy PL structures on M

Results on $S_{PL}(T^k \times D^n)$ imply:

$$|\pi_n(TOP/PL)| = \begin{cases} 0 & n \neq 3 \\ \leq 2 & n = 3 \end{cases}$$

$$\begin{tikzcd} & BPL \arrow[d] \\ M \arrow[r] \arrow[ur, dashed] & BTOP \end{tikzcd}$$

But if $\pi_3(TOP/PL) = 0$, then every map $M \longrightarrow BTOP$ lifts to BPL but you can use the Rokhlin invariant to show that $S^1 \times E_8$ does not admit a PL-structure.

Let K be a triangulation on top mfd M^n

$$c(K) = \sum_{\sigma \in K^{n-4}} [Lk(\sigma)]_\sigma \in H_{n-4}(M; \Theta^3_{\mathbb{Z}}) \cong H^4(M; \Theta^3_{\mathbb{Z}})$$

short exact sequence coming from Rokhlin invariant

$$0 \longrightarrow \ker \mu \longrightarrow \Theta^3_{\mathbb{Z}} \longrightarrow \mathbb{Z}/2 \longrightarrow 0 \quad (*)$$

and long exact sequence

$$\cdots \longrightarrow H^4(M; \Theta^3_{\mathbb{Z}}) \xrightarrow{\mu} H^4(M; \mathbb{Z}/2) \xrightarrow{\delta} H^5(M; \ker\mu) \longrightarrow \cdots \quad (**)$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \\ c(K) & \longmapsto & \Delta(M) \end{array}$$

i.e. $\mu(c(K)) = \Delta(M)$

$(**)$ tells us

M admits a triangulation $\Longrightarrow$ $\delta(\Delta(M)) = 0 \in H^5(M; \ker\mu)$

also true due to Galewski-Stern, Matumoto

They also showed that $(*)$ does not split $\Longleftrightarrow$ for every $n \geq 5$, $\exists\, M^n$ with $\delta(\Delta(M)) \neq 0$

Manolescu proved $(*)$ does not split

Can use Steenrod squares to give examples of non-triangulable top mfds:

s.e.s. $0 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathbb{Z}/4 \longrightarrow \mathbb{Z}/2 \longrightarrow 0$

connecting hom in l.e.s. is Sq^1

$$H^k(M; \mathbb{Z}/2) \xrightarrow{Sq^1} H^{k+1}(M; \mathbb{Z}/2)$$

Exercise: If $\dim M \geq 5$ and $Sq^1(\Delta(M)) \neq 0$ then $\delta(\Delta(M)) \neq 0$

Example: (Kronheimer)

Let X be a simply connected 4-mfd with intersection form

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and $\Delta(X) \neq 0$ (Exists due to Freedman)

Freedman also implies $\exists$ orientation reversing homeomorphism

$$f : X \to -X$$

$$M^5 := (X \times I) / (x,0) \sim (f(x), 0) \quad \text{mapping torus}$$

Exercise: $Sq^1(\Delta(M)) \neq 0$

Note: M is non-orientable; all non-triangulable 5-mfds are non-orientable

P^6 = circle bundle over M associated to oriented double cover

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Manolescu's invariant $\beta(Y) \in \mathbb{Z}$ defined using Pin(2)-equivariant Seiberg-Witten Floer homology

$$\mathbb{H} = \{ x + yi + zj + wk \mid x, y, z, w \in \mathbb{R} \} = \mathbb{C} \oplus \mathbb{C}j$$

quaternions

$ij = k$

unit quaternions $S(\mathbb{H}) = SU(2)$

$\hookrightarrow$ unitary matrices w/ $\det = 1$

$\hookrightarrow$ matrices of this form:

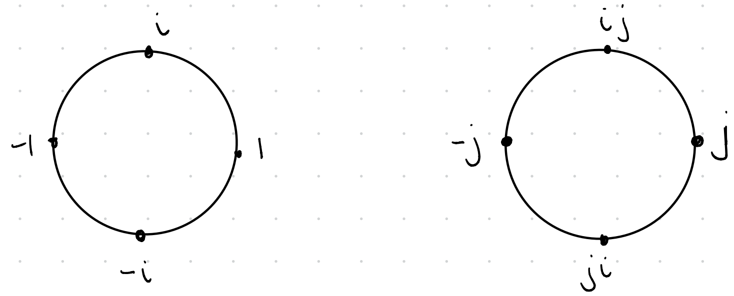
$$\left\{ \begin{pmatrix} a+bi & c+di \\ -c+di & a-bi \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d \in \mathbb{R} \\ a^2+b^2+c^2+d^2=1 \end{matrix} \right\}$$

$$S^1 = \mathbb{C} \cap S(\mathbb{H})$$

$$Pin(2) = S^1 \cup S^1_j \subset \mathbb{C} \cup \mathbb{C}j = \mathbb{H}$$

$j^2 = -1$

$ij = -ji$



S^1 $\qquad\qquad$ S^1_j

To a 3-mfd Y (with some extra data) one can associate a space I (up to homotopy). The space I admits an action by $Pin(2)$

$$SWFH_*^{Pin(2)}(Y) = Pin(2)\text{-equivariant homology of } I$$

$\downarrow$

Seiberg Witten Floer Homology

Equivariant (co)homology : Borel constructions

Goal: Define a homology theory for spaces with a group action

Let X be a space with a topological group G acting on it

$$X \circlearrowleft G$$

Example: $S^2 \circlearrowleft S^1$ by rotation

first guess: quotient S^2/S^1 = interval (contractible)
(action not free - fixes N. and S. pole)

Classifying space BG

EG weakly contractible space on which G acts properly and freely

$\downarrow$

BG

weakly contractible → all homotopy groups are trivial

→ for CW complex, this is when it is contractible

Example:

$G = \mathbb{Z}$ $\quad$ $EG = \mathbb{R}$
$\downarrow$
$BG = S^1$

Remark:
$H^*(BG)$ group cohomology of G

Example: $G = \mathbb{Z}/2$ $\quad$ $EG = S^\infty$
$\downarrow$
$BG = \mathbb{RP}^\infty$

Example: $G = S^1$ $\qquad$ $$\begin{array}{c} EG = S^\infty \\ \downarrow \\ BG = \mathbb{C}P^\infty \end{array}$$

$$H^*(\mathbb{C}P^\infty; \mathbb{Z}) = \mathbb{Z}[u]$$

$\deg u = 2$

Example: $G = SU(2)$ = unit quaternions

Exercise:

1. $BSU(2) = \mathbb{H}P^\infty$

2. Compute cohomology ring $H^*(\mathbb{H}P^\infty; \mathbb{Z})$

homotopy quotient:

$X \circlearrowleft G$

$$EG \times_G X = EG \times X / G$$

G acts on $EG \times X$ via diagonal direction

Action of G on $EG \times X$ is free since G acts freely on EG

$$p : EG \times_G X \longrightarrow EG/G = BG$$

So we have a bundle

$$\begin{tikzcd} X \arrow[r] & EG \times_G X \arrow[d] \\ & BG \end{tikzcd}$$

<u>Borel cohomology</u> or <u>equivariant cohomology</u> of $X^{\circlearrowleft G}$ is

$$H^*_G(X;R) := H^*(EG \times_G X ; R)$$

<u>Example:</u>

G trivial group

$$H^*_G(X;R) = H^*(X;R)$$

<u>Example:</u>

X contractible

$$H^*_G(X;R) = H^*(BG;R)$$

<u>Example:</u>

G acts freely on X

projection $EG \times_G X \xrightarrow[\text{equiv.}]{\text{homotopy}} X/G$

$$\Longrightarrow \quad H^*_G(X;R) = H^*(X/G;R)$$

<u>Note</u>: $p : EG \times_G X \longrightarrow EG/G = BG$

$$p^* : H^*(BG; R) \longrightarrow H^*(EG \times_G X; R) = H^*_G(X; R)$$

$$\Longrightarrow \quad H^*_G(X; R) \text{ is a } H^*(BG; R)\text{-module}$$

compose p^* with cup product

Also, $H^G_*(X; R)$ is a $H^*(BG; R)$-module

compose p^* with cap product

<u>Example</u>:

$S^2 \circlearrowleft S^1$ by rotation (along z-axis)

$$\begin{tikzcd} S^2 \arrow[r] & S^\infty \times_{S^1} S^1 \arrow[d] \\ & \mathbb{C}P^\infty \end{tikzcd}$$

given a fibration, there is a spectral sequence associated to it

Spectral sequence

$$\begin{array}{ccccc} H^0(\mathbb{C}P^\infty) & 0 & H^2(\mathbb{C}P^\infty) & 0 & H^4(\mathbb{C}P^\infty) \; \cdots \\ 0 & 0 & 0 & 0 & 0 \\ H^0(\mathbb{C}P^\infty) & 0 & H^2(\mathbb{C}P^\infty) & 0 & H^4(\mathbb{C}P^\infty) \; \cdots \end{array}$$

higher differentials $\quad d_{2+i} \quad$ $1+i$ down, $2+i$ right

see full computation, Tu Ch. 7 "Introductory lectures on equivariant cohomology"

Upshot: $SWFH_*^{Pin(2)}(Y)$ is a module over $H^*(BPin(2))$

Q: What is $H^*(BPin(2))$?

unit quaternions

$Pin(2) = S^1 \cup S^1_j \subset SU(2) \subset \mathbb{H}$

In fact, $\exists$ a fibration

$$\begin{tikzcd} Pin(2) \arrow[r] & SU(2) \arrow[d] \\ & \mathbb{RP}^2 \end{tikzcd}$$

$$\begin{tikzcd} S^1 \cup S^1_j \arrow[r] \arrow[d, "p"'] & \{ x + yi + zj + wk \mid \underbrace{x^2+y^2+z^2+w^2=1, \; x,y,z,w \in \mathbb{R}}_{\text{this is } S^3} \} \\ \mathbb{RP}^2 & \end{tikzcd}$$

Exercise:

projection map p is composition of the Hopf fibration map and antipodal map on S^2

fibration

$$\begin{tikzcd} \mathbb{RP}^2 \arrow[r] & BPin(2) \arrow[d] \\ & BSU(2) = \mathbb{HP}^\infty \end{tikzcd}$$

$$\begin{tikzcd} & ESU(2) \arrow[dl] \arrow[d] \\ BPin(2) \arrow[r] & BSU(2) \end{tikzcd}$$

Spectral sequence with $\mathbb{F} = \mathbb{Z}/2$ coefficients

$$\begin{array}{ccccccccc}
\mathbb{F} & 0 & 0 & 0 & \mathbb{F} & 0 & 0 & 0 & \mathbb{F} \\
\mathbb{F} & 0 & 0 & 0 & \mathbb{F} & 0 & 0 & 0 & \mathbb{F} \quad \cdots \\
\mathbb{F} & 0 & 0 & 0 & \mathbb{F} & 0 & 0 & 0 & \mathbb{F}
\end{array}$$

No room for higher differentials

$$H^*(BPin(2); \mathbb{F}) = \mathbb{F}[Q; V]/Q^3$$

$\deg Q = 1$

$\deg V = 4$