8803

Week 9

Monday pg 2

Wednesday pg 14

Last time:

Kirby-Seibennman invariant

M^n top mfd, $\Delta(M) \in H^4(M; \mathbb{Z}/2)$

$n \geq 5$ $\quad \Delta(M) = 0 \iff M$ admits PL structure

$n = 4$ $\quad \Delta(M) = 0 \Longleftarrow M$ admits PL structure

$$\begin{tikzcd} & BPL \arrow[d] \\ M \arrow[ur, dashed] \arrow[r] & BTOP \end{tikzcd}$$

Proposition (Kirby-Siebennman)

A top mfd M^n, $n \geq 5$, admits a PL structure

$\iff$

its top tangent bundle admits a PL structure

Equivariant (co-)homology: Borel construction

$$H^*_G(X; R) = H^*(EG \times_G X; R)$$

module over $H^*(BG; R)$

S^1-equivariant homology

$\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$

$$\begin{array}{ccc} S^1 & \longrightarrow & S^\infty \\ & & \downarrow \\ & & \mathbb{C}P^\infty = BS^1 \end{array}$$

$H^*(\mathbb{C}P^\infty; \mathbb{F}) = \mathbb{F}[u]$

$\deg u = 2$

$H_*^{S^1}(X; \mathbb{F})$ module over $\mathbb{F}[u]$

Recall: $\mathbb{F}[u]$ is a principal ideal domain for field $\mathbb{F}$

Any finitely generated module M over $\mathbb{F}[u]$ (non-canonically) isomorphic to

$$\underbrace{\bigoplus_{j=1}^{N} \mathbb{F}[u]}_{\text{free parts}} \oplus \underbrace{\bigoplus_{i=1}^{m} \mathbb{F}[u]/(p_i)}_{\text{torsion parts}}$$

ideal gen. by polynomials

Moreover, if M is graded, then each polynomial p_i must be homogeneously graded, ie.

$$p_i = u^{m_i} \quad \text{for some } m_i$$

Note: $u^2 + u + 1$ is not homogenous.

u^2 is.

Hence $M = \bigoplus_{j=1}^{N} \mathbb{F}_{d_j}[u] \oplus \bigoplus_{i=1}^{m} \mathbb{F}_{e_i}[u]/u^{m_i}$

where $\mathbb{F}_d[u]$ denotes $\mathbb{F}[u]$ where $gr\ 1 = d$.

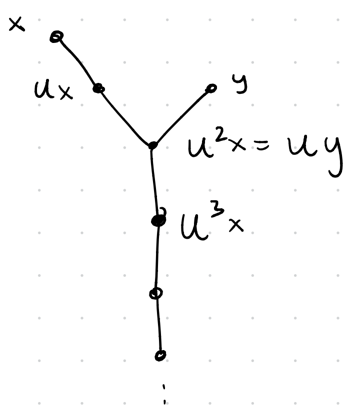
* Convention: to line up with Heegaard Floer conventions, from now, $\deg u = -2$

Suppose $N=1$. Then we can define

$$d(M) = \max\{ gr(x) \mid x \in M,\ u^k x \neq 0\ \forall\ k>0 \}$$

Example:

$\mathbb{F} = \mathbb{F}[u]/u$

$M =$ 

gr:
2
0
-2
-4
-6
⋮

$M \cong \mathbb{F}[u] \oplus \mathbb{F}$

$\cong \mathbb{F}[u]\langle x\rangle + \mathbb{F}[u]\langle ux+y\rangle$

For $k>0$

$u^k M$ is 1-dim

Pin(2)-equivariant homology

$$H^*(BPin(2); \mathbb{F}) = \mathbb{F}[Q,V]/Q^3$$

★ conventions: deg $Q = -1$

deg $V = -4$

Seiberg-Witten Floer homology — eats a 3 mfd and outputs Pin(2) equiv. homology

takeaway: this is a module over BG

$SWFH_*^{Pin(2)}(Y)$ module over $\mathbb{F}[Q,V]/Q^3$

Note: $\mathbb{F}[Q,V]/Q^3$ not a P.I.D.

eg. $\langle Q, V\rangle$ not principal

Manolescu proved that for $N \gg 0$

roughly picking out d invariant

$V^N \cdot SWFH_*^{Pin(2)}(Y)$ is 3-dim x_1, x_2, x_3

losing info on the torsion pieces

and $Q x_3 = x_2$

$Q^2 x_3 = Q x_2 = x_1$

Define:

$$A(Y) = \max \{ gr(x) \mid x \in SWFH_*^{Pin(2)}(Y), \text{ for } N \gg 0,\ V^N \cdot x \neq 0, \text{ and } V^N \cdot x \in \operatorname{Im} Q^2 \}$$

"x_1"

picking out the x_1 part because Q does not annihilate x_2 or x_3

$$B(Y) = \max \{ gr(x) \mid x \in SWFH_*^{Pin(2)}(Y) \text{ for } N \gg 0,\ V^N \cdot x \neq 0,\ QV^N \cdot x \neq 0,\ Q^2 V^N \cdot x = 0 \}$$

"x_2"

$$C(Y) = \max \{ gr(x) \mid x \in SWFH_*^{Pin(2)}(Y) \text{ for } N \gg 0,\ V^N \cdot x \neq 0,\ Q^2 V^N \cdot x \neq 0 \}$$

"x_3"

Renormalize: $\alpha = \frac{A}{2}$ $\quad \beta = \frac{B-1}{2}$ $\quad \gamma = \frac{C-2}{2}$

Theorem (Manolescu)

1. α, β, γ are invariants of homology cobordism
2. $\beta \bmod 2 =$ Rokhlin invariant
3. $\beta(-Y) = -\beta(Y)$

$SWFH_*^{Pin(2)}(Y)$ is closely related to involutive Heegaard Floer homology (Hendricks - Manolescu), a refinement of Heegaard Floer homology (Ozsváth - Szabó)

Heegaard Floer and knot Floer homology have many applications to homology cobordism and knot concordance.

Heegaard Floer homology

~ invariant of 3-mfds, output algebraic object given the "right kind" of input.

$$\begin{array}{ccccc} \text{Heegaard diagram } \mathcal{H} \text{ for } Y & \rightsquigarrow & \text{chain complex } CF^-(\mathcal{H}) & \rightsquigarrow & HF^-(\mathcal{H}) = H_*(CF^-(Y)) \end{array}$$

$CF^-(\mathcal{H})$: free, fin. gen. graded chain complex over $\mathbb{F}[u]$

$HF^-(\mathcal{H})$: fin. gen graded module over $\mathbb{F}[u]$, $\deg u = -2$

Remark: $HF^-(Y)$ isomorphic to S^1-equivariant $SWFH_*^{S^1}(Y)$ by work of Kutluhan-Lee-Taubes, Colin-Ghiggini-Honda, Manolescu-Lidman

$$HF^-(Y) = \bigoplus_{s \in spin^c(Y)} HF^-(Y, s)$$

Remark: $spin^c(Y) \xleftrightarrow{1:1} H_1(Y; \mathbb{Z}) \cong H^2(Y; \mathbb{Z})$

Osváth-Szabó showed that for Y a $\mathbb{Q}HS^3$

$$HF^-(Y, s) \cong \mathbb{F}[U] \oplus \bigoplus_{i=1}^{m} \mathbb{F}[U]/U^{n_i} \quad \forall\, s \in spin^c(Y)$$

Some say Heegaard Floer hom. is TQFT-like:

A cobordism $W: Y_0 \longrightarrow Y_1$ induces a module homomorphism $F_W: HF^-(Y_0) \longrightarrow HF^-(Y_1)$

Other Flavors:

s.e.s. $0 \longrightarrow \mathbb{F}[U] \xrightarrow{U} \mathbb{F}[U] \longrightarrow \mathbb{F} \longrightarrow 0$

$$0 \longrightarrow CF^-(\mathcal{H}) \xrightarrow[U]{\text{mult by}} CF^-(\mathcal{H}) \longrightarrow \widehat{CF}(\mathcal{H}) \longrightarrow 0$$

$\widehat{CF}(\mathcal{H})$ is obtained from $CF^-(\mathcal{H})$ by setting $U=0$.

* set $U=0$ first and then take homology

$$\widehat{HF}(Y) = H_*(\widehat{CF}(\mathcal{H}))$$

weaker than HF^- (but sometimes easier to work with)

Example:

$$CF^-(\mathcal{H}) = \langle x, y, z \rangle_{\mathbb{F}[U]}$$

$\partial x = 0$ $\quad$ $\ker \partial = \langle x, z \rangle$

$\partial y = Uz$ $\quad$ $\operatorname{im} \partial = \langle Uz \rangle$

$\partial z = 0$

$$H_*(CF^-(\mathcal{H})) \cong \mathbb{F}[U]\langle x \rangle \oplus \mathbb{F}\langle z \rangle$$

$$\widehat{CF}(\mathcal{H}) = \langle x, y, z \rangle_{\mathbb{F}}$$

differential is same but set $U = 0$ $\left\{ \begin{array}{l} \partial x = 0 \\ \partial y = 0 \\ \partial z = 0 \end{array} \right.$ $\quad$ $\ker \partial = \langle x, y, z \rangle$, $\operatorname{im} \partial = \langle 0 \rangle$

$$H_*(\widehat{CF}(\mathcal{H})) = \mathbb{F}^3$$

Exercise:

Show that $\widehat{HF}(Y)$ is determined by $HF^-(Y)$

(but not by just setting $U = 0$!)

s.e.s.

$$0 \longrightarrow \mathbb{F}[u] \xrightarrow{\text{inclusion}} \mathbb{F}[u,u^{-1}] \xrightarrow{\text{coker}} \mathbb{F}[u,u^{-1}] \big/ \mathbb{F}[u] \longrightarrow 0$$

$$0 \longrightarrow CF^-(\mathcal{H}) \longrightarrow \underbrace{CF^-(\mathcal{H}) \otimes_{\mathbb{F}[u]} \mathbb{F}[u,u^{-1}]}_{\text{call this } CF^\infty} \longrightarrow CF^+(\mathcal{H}) \longrightarrow 0$$

$$HF^+(Y) = H_*(CF^+(Y))$$

$$HF^\infty(Y) = H_*(CF^\infty(Y))$$

turns out that this invariant is fairly boring

if Y $\mathbb{Q}HS^3$, then $HF^\infty(Y, \mathfrak{s}) \cong \mathbb{F}[u,u^{-1}]$

Exercise: If Y $\mathbb{Q}HS^3$, show that $HF^+(Y)$ is determined by $HF^-(Y)$ (and vice versa)

contain the same amount of info. in contrast to $\widehat{HF}$ which contains strictly less.

Flavors are more useful in dif. contexts.

Notation: $CF^\circ(Y)$, $HF^\circ(Y)$ where $\circ = +, -, \wedge, \infty$

depends on choice of Heegaard diagram but same for Y.

Knots in 3-mfds:

A null-homologous knot K in Y induces a filtration on $CF^\circ(\mathcal{H})$

nested family of subcomplexes

$$\ldots \subseteq F_{i-1} \subseteq F_i \subseteq F_{i+1} \subseteq \ldots$$

The associated graded complex is

$$gCFK^\circ(Y,K) = \bigoplus_i F_i / F_{i-1}$$

"Hence" an abuse of notation

knot Floer homology

$$HFK^\circ(Y,K) = H_*\left(\bigoplus_i F_i / F_{i-1}\right)$$

filtration gives a 2^{nd} grading, so bigraded

m = homological (Maslov) grading

s = Alexander grading coming from filtration

$\widehat{HFK}(Y, K)$ with $\mathbb{F} = \mathbb{Z}/2$ coefficients is bigraded vector space

$\boxed{\text{Theorem}}$ (Ozsváth - Szabó)

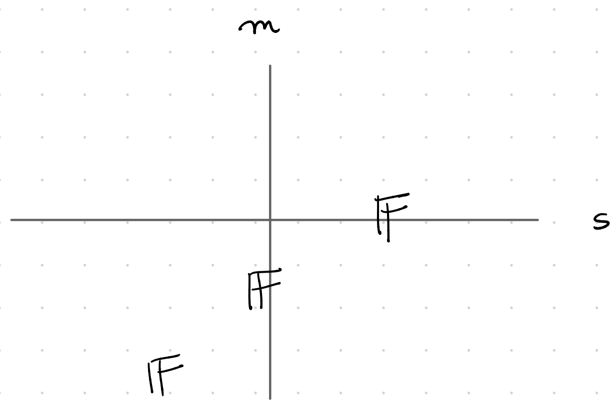
$\widehat{HFK}(S^3, K)$ categorifies $\Delta_K(t)$;

$$\Delta_K(t) = \sum_{m,s} (-1)^m t^s \dim \widehat{HFK}_m(K, s)$$

Example:

$K = T_{2,3}$

$\widehat{HFK}(T_{2,3})$



$$\Delta_{T_{2,3}}(t) = t^{-1} - 1 + t$$

Exercise:

$\deg \Delta_K(t) \leq g(K)$

symmetrized

$\deg \Delta_K(t) =$ highest positive power

Ex: $\deg(t - 1 + t^{-1}) = 1$

Theorem (Ozsváth-Szabó)

$\widehat{HFK}$ detects genus,

$$g(K) = \max\{ s \mid \widehat{HFK}(K,s) \neq 0 \}$$

tells you what genus of K is.

Last time:

Heegaard Floer homology $HF^\circ(Y)$

Knot Floer homology $HFK^\circ(Y)$

$\widehat{HFK}(Y)$ categorifies $\Delta_K(t)$

$$\Delta_K(t) = \sum_{m,s} (-1)^m t^s \dim \widehat{HFK}_m(K, s)$$

Alexander grading

homological grading

$$\deg \Delta_K(t) \leq g(K)$$

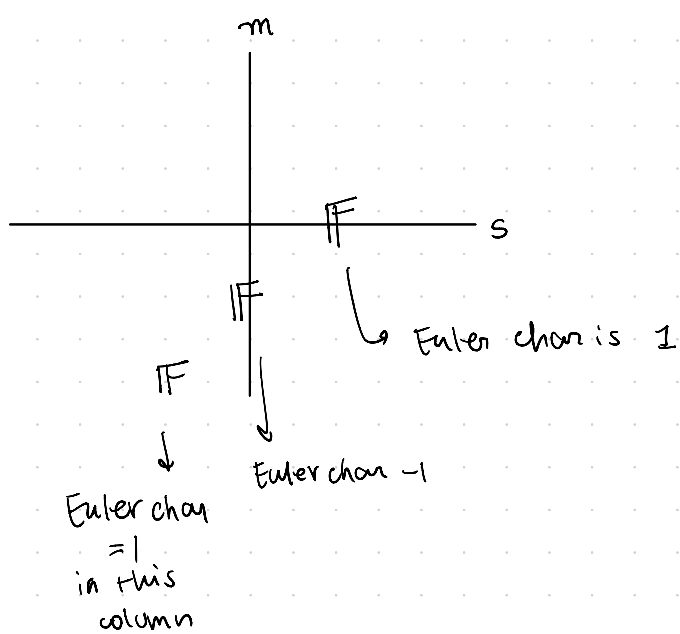
Theorem (Ozsváth-Szabó)

$\widehat{HFK}$ detects genus,

$$g(K) = \max\{s \mid \widehat{HFK}(K,s) \neq 0\}$$

Example:

$\widehat{HFK}(T_{2,3})$



$t^{-1} \quad -1 + t$

defn:

A knot $K \subset S^3$ is fibered if $S^3 - K$ is a fiber bundle over S^1

Exercise

If K is fibered, then $\Delta_K(t)$ is monic

Theorem: (Ghiggini, Ni)

$\widehat{HFK}$ detects fiberedness

K is fibered $\iff$ $\widehat{HFK}(K, g(K)) \cong \mathbb{F}$

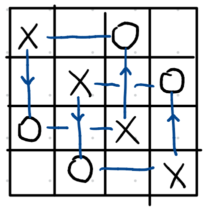
Grid Homology

→ Ozsváth-Stipsicz-Szabó "Grid homology for Knots and Links"

Defn: A planar grid diagram $\mathbb{G}$ is an $n \times n$ grid such that n squares are marked with X's and n squares are marked with O's such that

1. Each column has exactly one O and one X
2. Each row has exactly one O and one X
3. No square has both an O and an X

Example:



To get an oriented link:

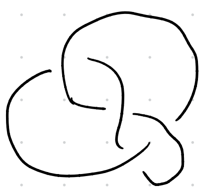
In each column connect X to O

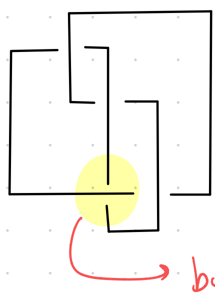
In each row connect O to X

s.t. vertical strands are ~~under~~crossings
over

Q: Can every link be represented by a grid diagram?

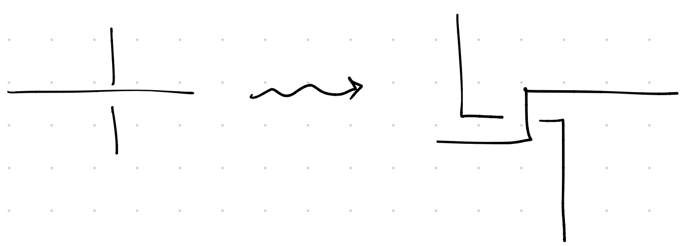
A: Yes





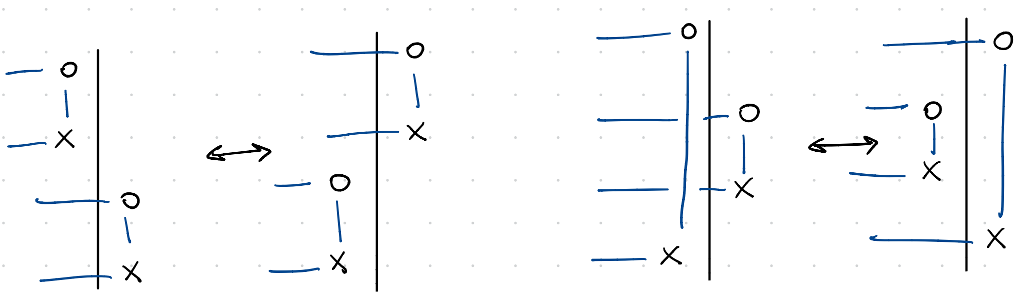
but not an over crossing

to fix this:

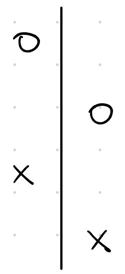


"Reidemeister moves" for grid diagrams: Grid moves

Column commutation:



Not allowed:

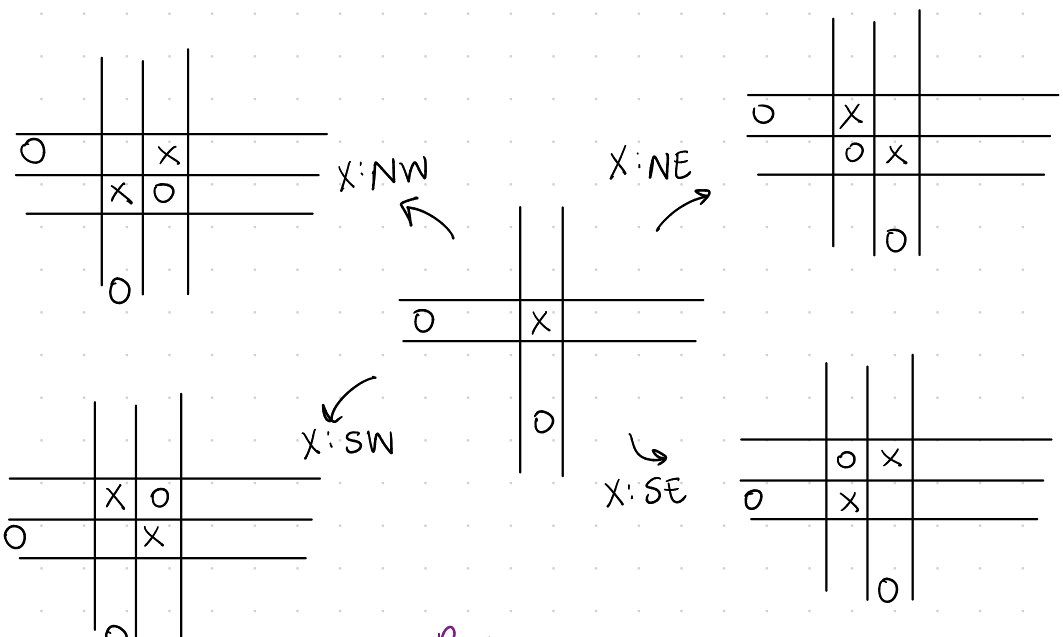


interleaved

Row commutation: defined analogously

Stabilization/destabilization:

$$n \times n \text{ grid} \underset{\text{destab}}{\overset{\text{stab}}{\rightleftharpoons}} (n+1)\times(n+1) \text{ grid}$$



Rmk: Can analogously de/stab along an O

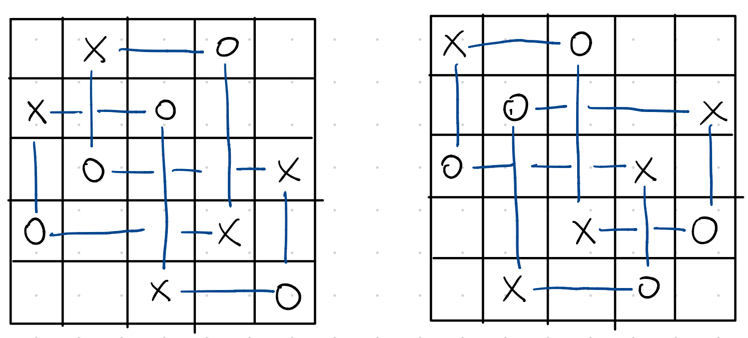
Exercise:

Verify that column/row commutation, de/stabilization do not change isotopy class of link

Theorem (Cromwell)

Two planar grid diagrams represent the same link $\Longleftrightarrow$ they can be related by a finite sequence sequence of commutation and de/stabilizations

Toroidal Grid Diagrams:



cyclic permutations of columns yields same knot

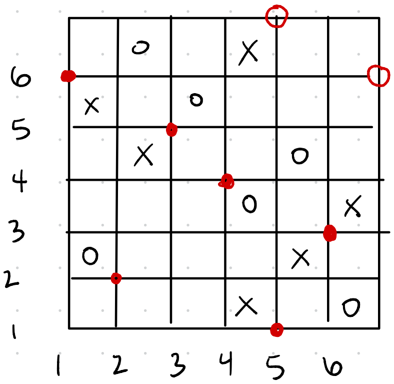
Similarly for cyclic permut. of rows.

Goal: Define a bigraded chain complex via. toroidal grid diagram $\mathbb{G}$ whose homology is a knot invariant, and whose graded Euler char is $\Delta_K(t)$

Grid states: $S(\mathbb{G})$ = bijections between vertical circles and horizontal circle

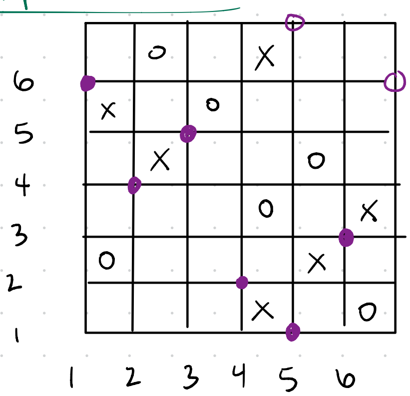
$= S_n$

Example 1:

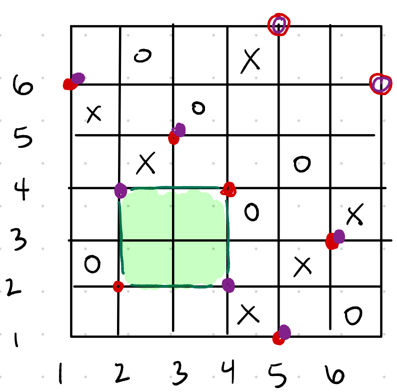


$$y = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 5 & 4 & 1 & 3 \end{pmatrix} \qquad \begin{pmatrix} \text{row no.} \\ \text{col. no.} \end{pmatrix}$$

Example 2:



$$z = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 4 & 5 & 2 & 1 & 3 \end{pmatrix}.$$

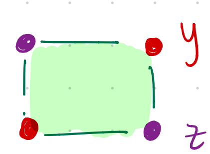


Example

Suppose $y, z \in S(G)$ s.t. y and z agree in exactly $n-2$ points

Consider the 2 points in y and the 2 points in z where they disagree.

Connect these 4 points to form a rectangle r



r goes from y to z

y's are in NE, SW corners, called initial corners

z's in NW, SE corners, called terminal corners

$y, z \in S(G)$

$\text{Rect}(y,z)$ = set of rectangles from y to z

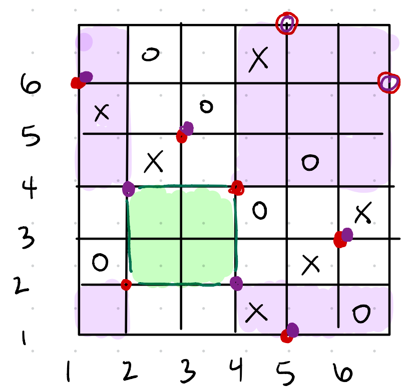
Note:

1.) $|\text{Rect}(y,z)| = \begin{cases} 2 & \text{if } y \text{ and } z \text{ agree in } n-2 \text{ points} \\ 0 & \text{otherwise} \end{cases}$

Recall toroidal, so "outside" rect as well

2.) Let $r \in \text{Rect}(y,z)$,

then $y \cap \text{Int}(r) = z \cap \text{Int}(r)$



Defn: A rectangle $r \in \text{Rect}(y,z)$ is empty if

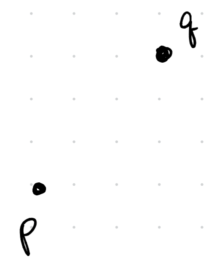
$$y \cap \text{Int}(r) = z \cap \text{Int}(r) = \emptyset$$

$\text{Rect}^{\circ}(y,z)$ = set of empty rectangles from y to z

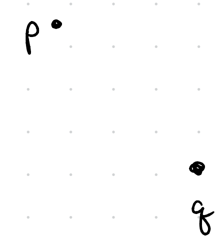
Bigrading on grid states

$p = (p_1, p_2)$ $\quad$ $q = (q_1, q_2)$

define $p < q$ if $p_1 < q_1$ and $p_2 < q_2$



$p < q$



$p \not< q \quad q \not< p$

Defn: Let P and Q be finite collections of points in $\mathbb{R}^2$

define
$$I(P, Q) = \#\{(p, q) \mid p \in P,\ q \in Q,\ p < q\}$$

$$J(P, Q) = \frac{I(P, Q) + I(Q, P)}{2}$$

$\mathbb{O}$ = set of O's
$\mathbb{X}$ = set of X's $\Big\}$ half integer coords

$y \in S(\mathbb{G})$ integer coords

fundamental domain $[0, n) \times [0, n)$

Defn: $$M_{O}(y) = J(y,y) - 2J(y,\mathbb{O}) + J(\mathbb{O},\mathbb{O}) + 1$$
$$= J(y-\mathbb{O},\, y-\mathbb{O}) + 1$$

$$M_{\mathbb{X}}(y) = J(y-\mathbb{X},\, y-\mathbb{X}) + 1$$

Defn: Maslov grading: $M(y) = M_{O}(y)$

Alexander grading: $A(y) = \frac{1}{2}\left(M_{O}(y) - M_{\mathbb{X}}(y)\right) - \frac{n-1}{2}$

Proposition

The functions $M: S(G) \longrightarrow \mathbb{Z}$, $A: S(G) \longrightarrow \mathbb{Z}$ are well-defined. Moreover, M is characterized by the following two properties:

1. Let y^{NWO} be the grid state consisting of the upper left corners of the O squares, then $M(y^{NWO}) = 0$
2. If $\mathrm{Rect}(y,z) \neq 0$, then $M(y) - M(z) = 1 - 2\#\{r \cap \mathbb{O}\} + 2\#\{y \cap \mathrm{Int}(r)\}$

Up to an overall additive constant, M is characterized by $r \in \mathrm{Rect}(y,z)$

$$A(y) - A(z) = \#\{r \cap \mathbb{X}\} - \#\{r \cap \mathbb{O}\}$$

The fully blocked grid chain complex:

$\widetilde{GC}(\mathbb{G})$ = bigraded chain complex over $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$ generated by $S(\mathbb{G})$

← indicates flavor   mod 2   no other point of y inside of them

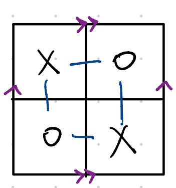
$$\tilde{\partial}_{\mathbb{O},\mathbb{X}}(y) = \sum_{z \in S(\mathbb{G})} \# \underbrace{\left\{ r \in \mathrm{Rect}^{\circ}(y,z) \;\middle|\; r \cap \mathbb{X} = r \cap \mathbb{O} = \emptyset \right\}}_{\text{shows up often. } \mathrm{Rect}^{\circ}_{\mathbb{O},\mathbb{X}}(y,z)} \cdot z$$

Exercise:

$\tilde{\partial}_{\mathbb{O},\mathbb{X}}$ lowers Maslov grading by 1 and preserves Alexander grading.

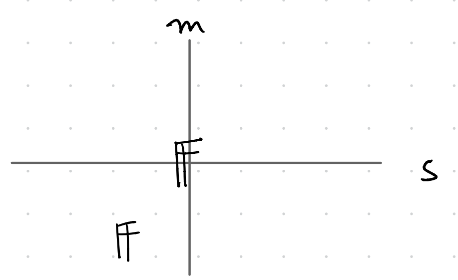
Define: $\widetilde{GH}(\mathbb{G}) = H_*(\widetilde{GC}(\mathbb{G}))$

Exercises:

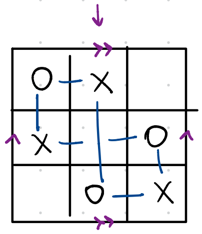
1. 

these are toroidal

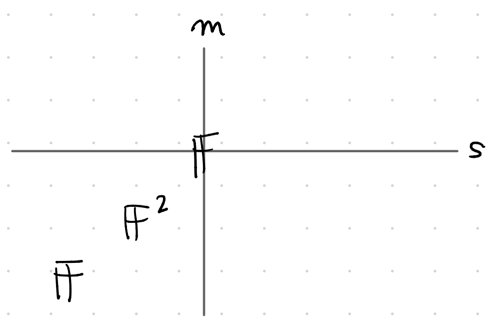
$\widetilde{GH}(\mathbb{G})$



$\widehat{GH}(\mathbb{G})$

2. 

Show that $\widetilde{GH}(\mathbb{G})$

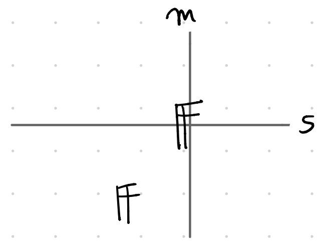


$\widetilde{GH}(\mathbb{G})$ is not a knot invariant!

However,

Theorem:

Let $\mathbb{G}$ be an $n \times n$ toroidal grid diagram for K.
Let W be the 2-dim vector space



Then $\exists$ a bigraded vector space $\widehat{GH}(\mathbb{G}) = \widehat{HFK}(K)$ s.t.

1. $\widetilde{GH}(\mathbb{G}) \cong \widehat{GH}(\mathbb{G}) \otimes W^{\otimes n-1}$

2. $\widehat{GH}(\mathbb{G})$ is a knot invariant