Problem Session from AMS Sectional Meeting in Baltimore, MD

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In what follows, all three-manifolds below are assumed to be closed and orientable (except in the last problem). All references, citations, and attributions are made to the best of our knowledge. When an attribution is omitted, we do not know the source or it is a well-known open problem. Any comments, corrections, updates, etc. are welcome.

1. (A. Moore)

   Problem 1 (Hedden)**.** *Do there exist hyperbolic three-manifolds Y_1 and Y_2 which are not genus 2 mutants, have the same hyperbolic volume, and have* $\operatorname{rk}\widehat{HF}(Y_1) = \operatorname{rk}\widehat{HF}(Y_2)$*?*

   With regards to mutation, it is a result of Ruberman that hyperbolic volume is preserved under mutation [Rub87]. The preservation of $\operatorname{rk}\widehat{HF}$ is currently unknown for mutation, although Clarkson shows that the rank of the Heegaard Floer homology coming from non-torsion Spin^c structures need not be preserved [Cla].

2. (C. Scaduto) Kronheimer-Mrowka have conjectured that for a three-manifold Y, $\widehat{HF}(Y)$ and $I^{\#}(Y)$ are isomorphic with rational coefficients [KM10]. It is known that $I^{\#}(Y)$ is $\mathbb{Z}/4$-graded [KM10] and $\widehat{HF}(Y)$ is $\mathbb{Z}/2$-graded [OS04c]. Note that for, say, integer homology spheres, this $\mathbb{Z}/2$-grading lifts to a $\mathbb{Z}$-grading.

   Problem 2 (Scaduto)**.** *Does there exist a $\mathbb{Z}/4$-lift of the $\mathbb{Z}/2$-grading on Heegaard Floer homology?*

3. (S. Sivek) Let ξ be a contact structure on Y^3. There are associated contact invariants in Heegaard Floer homology: $c(\xi) \in \widehat{HF}(-Y)$ and $c^+(\xi) \in HF^+(-Y)$ [OS05a]; the natural map from $\widehat{HF}$ to HF^+ takes $c(\xi)$ to $c^+(\xi)$. It is a result of Ghiggini that if (Y, ξ) is strongly symplectically fillable, then $c(\xi) \neq 0$ [Ghi06]. This is proved by showing that $c^+(\xi) \neq 0$. On the other hand, Ozsváth-Szabó prove directly (not using $c^+(\xi)$) that if (Y, ξ) is Stein fillable, then $c(\xi) \neq 0$ [OS05a].

   Problem 3 (Sivek)**.** *Is there a proof that $c(\xi) \neq 0$ for strongly symplectically fillable contact structures which only makes use of $\widehat{HF}$?*

   This is motivation for the following.

Conjecture 1 (Baldwin-Sivek [BS]). *Let Y^3 be an integer homology sphere. If (Y,ξ) is Stein fillable by (X^4,J), where $c_1(J)\neq 0$, then there exists a non-trivial representation $\pi_1(Y)\to SU(2)$.*

4. (T. Lidman) In light of Conjecture 1, it's natural to ask if the Stein fillability condition is necessary.

   Problem 4. *If Y is an integer homology three-sphere other than S^3, does there exists a non-trivial representation $\pi_1(Y)\to SU(2)$?*

   If Y is ± 1-surgery on a non-trivial knot K in S^3, then such a representation is guaranteed by Kronheimer-Mrowka [KM04]. This includes the Poincaré homology sphere $\Sigma(2,3,5)$, which can also be seen to have such a representation since it has non-trivial Casson invariant. If Y is a Seifert fibered or graph manifold homology sphere other than $\Sigma(2,3,5)$, it admits a co-orientable taut foliation by [BRW05] and [BB] respectively; it thus has a non-trivial $SU(2)$ representation by [KM04]. It is conjectured in [LR] that if $\alpha\in(-1,1)$, then $S^3_\alpha(K)$ admits a co-orientable taut foliation. Thus, if that conjecture were true, the answer to Problem 4 would be yes for all homology spheres obtained by surgery on a knot in S^3.

5. (T. Lidman)

   Problem 5. *Are there rational homology spheres that have torsion in HF^+? Are there any three-manifolds with torsion in $\widehat{HF}$? Are there knots with torsion in their knot Floer homology?*

   The first known instance of torsion in Heegaard Floer homology comes from Jabuka-Mark, who show that for a closed, orientable surface Σ_g, the group $HF^+(\Sigma_g\times S^1)$ has torsion for $g\geq 3$ [JM08]. That one can construct n-torsion for any n follows from a computation of Kronheimer-Mrowka [KM07] together with the isomorphism to Heegaard Floer homology due to Kutluhan, Lee, and Taubes [KLTa, KLTb, KLTc, KLTd, KLTe] or Taubes [Tau10a, Tau10b, Tau10c, Tau10d, Tau10e] and Colin, Ghiggini, and Honda [CGHa, CGHb, CGHc].

6. (T. Lidman) Recall that a group G is *left-orderable* if there exists a left-invariant, strict total order on G.

   Conjecture 2 (Boyer-Gordon-Watson [BGW13]). *If Y is an irreducible rational homology sphere, then $\pi_1(Y)$ is left-orderable if and only if Y is not an L-space (i.e., if $\operatorname{rk}\widehat{HF}(Y)>|H_1(Y;\mathbb{Z})|$).*

   Both of these notions are related to taut foliations. It follows from [OS04a] that an irreducible manifold with a co-orientable taut foliation has non-trivial reduced Floer homology. In particular, if Y is a rational homology sphere with a co-orientable taut foliation, then Y is not an L-space. Given a co-orientable taut foliation with hyperbolic leaves, Thurston's universal circle construction constructs an orientation-preserving action of $\pi_1(Y)$ on S^1. In many cases, this can be lifted to an action on $\mathbb{R}$. Since $\mathrm{Homeo}_+(\mathbb{R})$ is a left-orderable group, the fundamental group of Y will be as well in this case. Such a lift is guaranteed, for instance, if Y is an integer homology sphere [BB, CD03].

Problem 6. *Are either of these equivalent to the existence of a co-orientable taut foliation on Y?*

All three of these notions are known to be equivalent for Seifert fibered spaces [BRW05], graph manifold integer homology spheres [BB], and branched double-covers of non-split alternating links [BGW13], as well as many other families of manifolds.

7. (A. Moore)

Problem 7 (A. Moore)**.** *Is left-orderability of the fundamental group of Y^3 invariant under symmetric surface mutation?*

In light of Conjecture 2, this is related to the question as to whether the rank of $\widehat{HF}$ is preserved under mutation for rational homology spheres (not discussed here).

It's also natural to ask about how Floer theory for knots behaves under the presence of certain surfaces. Recall that a knot is *n-string prime* if there does not exist an essential $2n$-punctured sphere in $S^3 \setminus K$.

Conjecture 3 (Moore)**.** *L-space knots are n-string prime.*

This is related to the question of whether L-space knots have any non-trivial mutations. The answer would be no if Conjecture 3 is true.

Similarly, one can ask about the relationship between knot Floer homology and Conway mutation. While it is known that knot Floer homology as a bigraded object is not preserved under mutation [OS04d], there are still other pieces of information that could be preserved.

Question 1 (Baldwin-Levine [BL12])**.** *Is δ-graded $\widehat{HFK}$ invariant under mutation?*

A weaker question: Is the total rank of $\widehat{HFK}$ invariant under Conway mutation? What about genus 2 mutation?

8. (M. Hogancamp)

Problem 8 (Hogancamp)**.** *Kronheimer-Mrowka proved that there is a spectral sequence from the reduced Khovanov homology of a link to the singular instanton link Floer homology [KM11]. Is this spectral sequence functorial? In other words, do cobordisms induce maps of spectral sequences? What about the spectral sequence from the reduced Khovanov homology of a link to the Heegaard Floer homology of the branched double-cover of the mirror?*

For some partial progress on this question see [LZ]. Also, Szabó has conjectured a combinatorial model for the spectral sequences from $\widetilde{\mathrm{Kh}}(L)$ to $\widehat{HF}(\Sigma(\overline{L}))$ [OS05b]. It is maybe an easier problem to establish functoriality for this spectral sequence.

9. (M. Hogancamp)

Problem 9. *Categorify Reshetikhin-Turaev invariants and/or Turaev-Viro invariants.*

10. (K. Hendricks) Let $\tau(K)$ denote the Ozsváth-Szabó/Rasmussen concordance invariant coming from knot Floer homology [OS04b, Ras03].

Problem 10 (Hendricks)**.** *Let K be a knot in S^3 and denote by $\tilde{K}$ the induced knot in the branched double-cover of K. Is there a relationship between $\tau(K)$ and some suitably-defined τ-invariants of $\tilde{K}$?*

It may be easiest to consider knots with determinant equal to one first.

11. (N. Zufelt)

Problem 11. *When is a three-manifold manifold surgery on a knot in S^3? That is, what is the link of S^3 in the big Dehn surgery graph?*

Some useful obstructions to being surgery on a knot are

- $H_1(S^3_{p/q}(K)) \cong \mathbb{Z}/p$.
- If $S^3_{p/q}(K)$ is not prime, there is a non-trivial lens space summand [GL89]. Therefore, for instance, $\Sigma(2,3,5)\#\Sigma(2,3,5)$ (or any other non-prime homology sphere) is not surgery on a knot in S^3.
- If $\pi_1(Y)$ is not normally generated, Y is not surgery on a knot.

There are other invariants that have been used to obstruct manifolds from being surgery on a knot, such as the Casson-Walker invariant [BL90], Taubes' periodic ends theorem [Auc97], and the correction terms from Heegaard Floer homology [Doia, Doib, HW].

12. (A. Moore) Recall that a knot is *strongly invertible* if there exists an orientation-preserving involution of S^3 which fixes the knot setwise and reverses orientation.

Problem 12 (Watson)**.** *Are L-space knots strongly invertible?*

For example, all tunnel number one knots are strongly invertible; this includes all torus knots. However, L-space knots can have arbitrarily large tunnel number (for example, by cabling). As far as we are aware, all known L-space knots are strongly invertible.

13. (T. Lidman) It is known that $T_{2,3}$ is the only genus one L-space knot [Ghi08].

Problem 13. *Is $T_{2,5}$ the only genus two L-space knot? This would imply that $\widehat{HFK}$ detects $T_{2,5}$. More generally, $\widehat{HFK}$ is known to detect the unknot, both trefoils, and the figure eight knot [OS04a, Ghi08]. Are there other knots which are detected by $\widehat{HFK}$?*

14. (T. Lidman) Let M be a bordered three-manifold with connected boundary. Let $\widehat{CFD}(M)$ denote the bordered type D invariant of M [LOTa]. Then work of Lipshitz-Ozsváth-Thurston shows that $\widehat{HF}(D(M)) \cong H_*(\mathrm{End}(\widehat{CFD}(M)))$, where $D(M)$ denotes the double of M [LOTb].

Problem 14 (Lipshitz)**.** *There is an obvious ring structure on $H_*(\mathrm{End}(\widehat{CFD}(M))$ given by composition. Does this ring structure contain any new information?*

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