INTRO TO HIGHER MATH
HOMEWORK 12 DUE DECEMBER 5

Prove the following, using the formal definitions of $\mathbf{Z}$ and $\mathbf{Q}$ from class. Recall that $\mathbf{Z} = (\mathbb{N} \times \mathbb{N})/\sim$ where

$$(m_1, n_1) \sim (m_2, n_2) \quad \text{if and only if} \quad m_1 + n_2 = m_2 + n_1,$$

and $\mathbf{Q} = (\mathbb{Z} \times \mathbb{N}^+)/\sim$ where

$$(a, b) \sim (c, d) \quad \text{if and only if} \quad a \cdot d = b \cdot c.$$

(1) Prove that addition and multiplication on $\mathbf{Z}$ are well-defined.
(2) Prove that $\leq$ on $\mathbf{Z}$ is a linear ordering
(3) Prove that addition and multiplication on $\mathbf{Q}$ are associative, commutative, and distributive.
(4) (a) Prove that $[(1, 1)]$ is the multiplicative identity in $\mathbf{Q}$. That is, show that $[(p, q)] \cdot [(1, 1)] = [(p, q)]$ for all $[(p, q)] \in \mathbf{Q}$.
   (b) Let $[(a, b)] \in \mathbf{Q}$ such that $a \neq 0$. Prove that $[(a, b)]$ has a multiplicative inverse in $\mathbf{Q}$, that is, that there is some $[(c, d)] \in \mathbf{Q}$ such that $[(a, b)] \cdot [(c, d)] = [(1, 1)]$.